Elliptic curves maximal over finite extensions

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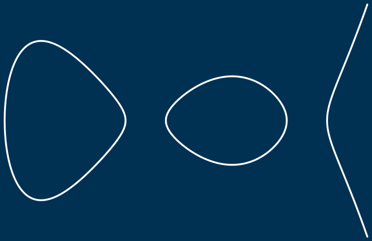
The arithmetic of maximal curves, the Hesse pencil and the Mestre curve



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Outline

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4. Ordinary case
5. Maximal over cubic extensions

Question

Given an elliptic curve E over a finite field k of cardinality q, does there exist a finite extension l of k such that the number of points on E over l is maximal with respect to the Hasse bound?

Finite fields

- A field k is *finite* if $|k|$ is finite.
- The characteristic of k is the (unique) prime p such that

$$\mathbb{Z}/p\mathbb{Z} \longrightarrow k, \qquad \bar{n} \longmapsto \sum_{i=1}^{n} 1$$

  is an injective ring homomorphism.
- $q = |k| = p^d$.
- Up to isomorphism, there is a unique field with q elements,
  - denote by $\mathbb{F}_q$,
  - $a^q = a$ for all $a \in \mathbb{F}_q$,
  - $(a+b)^p = a^p + b^p$ for all $a, b \in \mathbb{F}_q$.
- If k is a subfield of l, then
  - l is an *extension* of k denoted by l/k,
  - the degree $[l : k] := \dim_k(l)$.

Algebraic curves

- Let k be a field.
- An *(algebraic) variety* C over k is – loosely speaking – an irreducible topological space such that locally C is the set of zeros in $\bar{k}^n$ of

$$F_1(x_1, \ldots, x_n) = 0, \quad \ldots, \quad F_r(x_1, \ldots, x_n) = 0,$$

  where $F_i \in k[X_1, \ldots, X_n]$.
- If l/k, then

$$C(l) := \{(x_1, \ldots, x_n) \in l^n : F_i(x_1, \ldots, x_n) = 0 \text{ for all } i\}.$$

- A *curve* over k is a 1-dimensional variety over k,
  - for example: $x^2 + y^2 = 1$.

Hasse-Weil-Serre bound

Theorem

If C is a non-singular of genus g over $\mathbb{F}_q$, then

$$q + 1 - g\lfloor 2\sqrt{q} \rfloor \leq |C(\mathbb{F}_q)| \leq q + 1 + g\lfloor 2\sqrt{q} \rfloor.$$

- A curve C of genus g over $\mathbb{F}_q$ is *maximal* if

$$|C(\mathbb{F}_q)| = q + 1 + g\lfloor 2\sqrt{q} \rfloor.$$

- Consider

$$N_q(g) := \max\{|C(\mathbb{F}_q)| : C \text{ a curve of genus } g \text{ over } \mathbb{F}_q\}.$$

- A typical construction of a curve with many points is:
  1. Let D be a curve (of lower genus) with many points.
  2. Consider curves C with a morphism $C \to D$.

Elliptic curves

- An *elliptic curve* (E, O) over k is a non-singular curve of genus 1 over k and a point $O \in E(k)$:
  - The zero set of a (short) *Weierstrass* equation $(\mathrm{char}\,(k) \neq 2, 3)$

    $$E(\bar{k}) = \{(x, y) \in \bar{k}^2 : y^2 = x^3 + ax + b\} \cup \{O\}$$

    with $a, b \in k$ such that $4a^3 + 27b^2 \neq 0$.
- $E(k)$ is a group with identity O.

Example

- Consider $\mathbb{F}_3 = \{\bar{0}, \bar{1}, \bar{2}\}$ and the elliptic curve E over $\mathbb{F}_3$

$$y^2 = x^3 - x.$$

- Since $a^3 = a$ for all $a \in \mathbb{F}_3$,

$$E(\mathbb{F}_3) = \{O, (\bar{0}, \bar{0}), (\bar{1}, \bar{0}), (\bar{2}, \bar{0})\}.$$

- Hence $|E(\mathbb{F}_3)| = 4 < 7 = 3 + 1 + \lfloor 2\sqrt{3} \rfloor$.
- Consider $\mathbb{F}_9 \cong \mathbb{F}_3(i) = \{a + bi : a, b \in \mathbb{F}_3\}$ with $i^2 + \bar{1} = \bar{0}$.
- Since $(a + bi)^3 = a^3 + b^3 i^3 = a - bi$ for all $a, b \in \mathbb{F}_3$,

$$\begin{aligned} E(\mathbb{F}_9) = \{O, &(\bar{0}, \bar{0}), (\bar{1}, \bar{0}), (\bar{2}, \bar{0}), \\ &(i, \pm(\bar{1} - i)), (\bar{1} + i, \pm(\bar{1} - i)), (\bar{2} + i, \pm(\bar{1} - i)), \\ &(-i, \pm(\bar{1} + i)), (\bar{1} - i, \pm(\bar{1} + i)), (\bar{2} - i, \pm(\bar{1} + i))\}. \end{aligned}$$

- Hence $|E(\mathbb{F}_9)| = 16 = 9 + 1 + \lfloor 2\sqrt{9} \rfloor$.

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Eigenvalues of Frobenius

Theorem

Let E be an elliptic curve over $\mathbb{F}_q$. Then

$$|E(\mathbb{F}_{q^n})| = q^n + 1 - a_n, \qquad a_n = \alpha^n + \bar{\alpha}^n$$

for all $n \in \mathbb{Z}_{>0}$, where $\alpha \in \mathbb{C}$ is a root of

$$X^2 - a_1 X + q.$$

- Choose α such that $\operatorname{Im}(\alpha) \geq 0$.
- $|\alpha| = \sqrt{q}$.
- $|a_n| \leq 2\sqrt{q}^n$.
- A recurrence relation

$$\begin{aligned} a_{n+1} &= \alpha(\alpha^n + \bar{\alpha}^n) + \bar{\alpha}(\alpha^n + \bar{\alpha}^n) - q\bar{\alpha}^{n-1} - q\alpha^{n-1} \\ &= a_1 a_n - q a_{n-1}. \end{aligned}$$

Isogenies

Let E and E' be elliptic curves over k.

- An *isogeny* $\phi : E \to E'$ over k is a morphism over k such that $\phi(O) = O'$.
- The curves are *isogeneous* over k if there is a non-constant isogeny $E \to E'$ over k.
  - This is an equivalence relation.
  - This is a weaker version of an isomorphism.
- If $k = \mathbb{F}_q$, then

$$E \text{ and } E' \text{ are isogeneous over } \mathbb{F}_q \iff |E(\mathbb{F}_q)| = \left|E'(\mathbb{F}_q)\right|.$$

Theorem (Waterhouse)

An isogeny class of elliptic curves over $\mathbb{F}_q$ corresponds to an integer a_1 such that $|a_1| \leq 2\sqrt{q}$ and some additional conditions.

Reformulated question

Question

Given a prime power q and an $a_1 \in \mathbb{Z}$ such that $|a_1| \leq 2\sqrt{q}$, does there exist a $n \in \mathbb{Z}_{>0}$ such that $-a_n = \lfloor 2\sqrt{q}^n \rfloor$.

Recall that

- $\alpha \in \mathbb{C}$ is a root of $X^2 - a_1 X + q$,
- $a_n = \alpha^n + \bar{\alpha}^n$.

Define $\beta = \frac{\alpha}{\sqrt{q}}$.

An important lemma

Lemma

$$-a_n = \lfloor 2\sqrt{q}^n \rfloor \quad \iff \quad |\beta^n + 1| \leq \frac{1}{\sqrt[4]{q}^n}.$$

Proof.

Since $|a_n| \leq 2\sqrt{q}^n$,

$$-a_n = \lfloor 2\sqrt{q}^n \rfloor \iff 0 \leq a_n + 2\sqrt{q}^n < 1 \iff |a_n + 2\sqrt{q}^n| < 1.$$

Use

$$\begin{aligned} a_n + 2\sqrt{q}^n = \alpha^n + \bar{\alpha}^n + 2\sqrt{q}^n &= \bar{\alpha}^n \left(\frac{\alpha^{2n}}{\sqrt{q}^{2n}} + 1 + 2\frac{\alpha^n}{\sqrt{q}^n} \right) \\ &= \bar{\alpha}^n (\beta^n + 1)^2. \end{aligned}$$

$\square$

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Supersingular elliptic curves

- An elliptic curve E over $\mathbb{F}_q$ is *supersingular* if $\gcd(a_1, q) \neq 1$.
- The pair q, a_1 is *supersingular* if β is a root of unity, that is $\beta^m = 1$ for some non-zero $m \in \mathbb{Z}$.

Proposition

If the pair q, a_1 is supersingular, then $-a_n = \left\lfloor 2\sqrt{q}^n \right\rfloor$ for some $n \in \mathbb{Z}_{>0}$ if and only if

$$a_1 \in \left\{0, \sqrt{q}, \pm\sqrt{2q}, \pm\sqrt{3q}, -2\sqrt{q}\right\}.$$

Moreover if such an n exists, then there exist infinitely many.

- The case q a square was already solved by Peter Doetjes.

Proof

First step:

- β is a root of $X^2 - \frac{a_1}{\sqrt{q}}X + 1$ and therefore also of

$$X^4 + \left(2 - \frac{a_1^2}{q}\right)X^2 + 1.$$

- $[\mathbb{Q}(\beta) : \mathbb{Q}] \in \{1, 2, 4\}$.
- If β is a root of unity of order n, then
  - $[\mathbb{Q}(\beta) : \mathbb{Q}] = \phi(n)$,
  - evaluate the above polynomials in $\zeta_n = e^{\frac{2\pi}{n}i}$ to get a_1.
- If $a_1 \in \left\{0, \pm\sqrt{q}, \pm\sqrt{2q}, \pm\sqrt{3q}, \pm 2\sqrt{q}\right\}$, then one of the above polynomials is a (product of) cyclotomic polynomials.

d	n	a_1	Φ_n
1	1	$2\sqrt{q}$	$X-1$
	2	$-2\sqrt{q}$	$X+1$
2	3	$-\sqrt{q}$	X^2+X+1
	4	0	X^2+1
	6	$\sqrt{q}$	X^2-X+1
4	5		$X^4+X^3+X^2+X+1$
	8	$\pm\sqrt{2q}$	X^4+1
	10		$X^4-X^3+X^2-X+1$
	12	$\pm\sqrt{3q}$	X^4-X^2+1

Last step:

- If the order n of β is even, then for $n' = \frac{n}{2}$

$$0 = \left|\beta^{n'} + 1\right| < \frac{1}{\sqrt[4]{q}^{n'}}.$$

  Hence $-a_{n'} = \left\lfloor 2\sqrt{q}^{n'} \right\rfloor$.

- If the order n of β is odd, that is $n = 1$ or $n = 3$, then

$$\left|\beta^{n'} + 1\right| \geq 1 > \frac{1}{\sqrt[4]{q}^{n'}}$$

  for all $n' \in \mathbb{Z}_{>0}$. Hence $-a_{n'} \neq \left\lfloor 2\sqrt{q}^{n'} \right\rfloor$.

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Ordinary elliptic curves

- An elliptic curve E is *ordinary* if $\gcd(a_1, q) = 1$.
- The pair q, a_1 is *ordinary* if β is not a root of unity.

Proposition

If the pair q, a_1 is ordinary, then $-a_n = \left\lfloor 2\sqrt{q}^n \right\rfloor$ for at most finitely many $n \in \mathbb{Z}_{>0}$. Furthermore q is not a square and n is odd.

- In this case, if $-a_n = \left\lfloor 2\sqrt{q}^n \right\rfloor$, then

$$0 < |\beta^n + 1| < \frac{1}{\sqrt[4]{q}^n}.$$

Linear forms in logarithms

- A *linear form in logarithms* is

$$m_1 \log(\gamma_1) + \ldots + m_r \log(\gamma_r)$$

with $m_i \in \mathbb{Z}$ and γ_i non-zero algebraic numbers (over $\mathbb{Q}$).

Theorem (Baker)

If $\log(\gamma_1), \ldots, \log(\gamma_r)$ *are linear independent over* $\mathbb{Q}$*, then*

$$\log|m_1 \log(\gamma_1) + \ldots + m_r \log(\gamma_r)| > -C \log \max\{|m_1|, \ldots, |m_r|\}$$

with C *a constant depending on* γ_i*.*

Corollary

If γ is an algebraic number such that $|\gamma| = 1$ and γ is not a root of unity, then

$$\log|\log(-\gamma^n)| > -(32d)^{400}\log(4)\log\log(4)\log(h)\log(n)$$

for all integers $n \geq 4$, where $d = [\mathbb{Q}(\gamma) : \mathbb{Q}]$ and $h \in \mathbb{Z}_{\geq 4}$ is an upper bound on the height of γ.

Sketch of proof.

For some $k \in \mathbb{Z}$

$$\log(-\gamma^n) = \log(-1) + n\log(\gamma) + 2\pi k i = (2k+1)\log(-1) + n\log(\gamma)$$

Since γ is not a root of unity, $|\log(\gamma)| < \pi$ and $|\log(-\gamma^n)| < \pi$.

$$\begin{aligned} |2k+1|\pi &= |\log(-\gamma^n) - n\log(\gamma)| \\ &\leq |\log(-\gamma^n)| + n|\log(\gamma)| < (n+1)\pi. \end{aligned}$$

Upper bound on the degree n

Let q, a_1 be ordinary and $-a_n = \lfloor 2\sqrt{q}^n \rfloor$ for some $n \in \mathbb{Z}_{\geq 4}$.

- Recall that
$$0 < |\beta^n + 1| < \frac{1}{\sqrt[4]{q}^n}.$$
- Observe that for all $|z| < c < 1$
$$|\log(1-z)| = \left|\sum_{k=1}^{\infty} \frac{z^k}{k}\right| < \sum_{k=1}^{\infty} c^k = \frac{c}{1-c}.$$
- Take $z = \beta^n + 1$ and $c = \frac{1}{\sqrt[4]{q}^n}$. Then
$$\log|\log(-\beta^n)| < -\log(\sqrt[4]{q}^n - 1)$$
- Baker's Theorem gives
$$-\tilde{C}\log(2q)\log(n) < \log|\log(-\beta^n)|$$
with $\tilde{C} = 2^{2800}\log(4)\log\log(4)$.

Convergents

- Recall that $a_1 = \alpha + \bar{\alpha}$ and $\alpha = e^{i\theta}$ with $\theta \in [0, \pi]$.

Proposition (Doetjes)

If $-a_n = \left\lfloor 2\sqrt{q}^n \right\rfloor$ *for some* $n \in \mathbb{Z}_{>0}$, *then*

$$\left| \frac{\theta}{\pi} - \frac{m}{n} \right| < \frac{1}{\pi} \sqrt{\frac{48}{48 - \pi^2}} \frac{1}{n \sqrt[4]{q}^n}$$

with m *an odd integer.*

- If $-a_n = \left\lfloor 2\sqrt{q}^n \right\rfloor$ for some $n \geq 3$ and either $q \geq 3$ or $n \geq 13$, then $\frac{m}{n}$ is a convergent of $\frac{\theta}{\pi}$ for some odd m.

Computer experiment

Compute ordinary triples q, a_1, n with $n > 1$ such that

$$-a_n = \lfloor 2\sqrt{q}^n \rfloor.$$

- Case $n = 3$ and $q < 10^3$:

q	a_1	q	a_1	q	a_1	q	a_1	q	a_1	q	a_1
2	1	37	6	103	10	229	15	479	22	787	28
3	2	47	7	167	13	257	16	487	22	839	29
5	2	61	8	173	13	293	17	571	24	967	31
8	3	67	8	193	14	359	19	577	24		
11	3	79	9	197	14	397	20	673	26		
17	4	83	9	199	14	401	20	677	26		
23	5	97	10	223	15	439	21	727	27		
27	5	101	10	227	15	443	21	733	27		

- Case $n = 5$ and $q < 10^6$:

q	a_1	q	a_1
2	-1	8807	-58
3	-1	10391	-63
11	-2	10399	165
23	-3	22159	-92
31	9	122147	-216
128	-7	192271	-271
317	-11	842321	1485
2851	-33		

- Case $n = 7$ and $q < 10^6$:

$$q = 5, \quad a_1 = 1.$$

- Case $n = 13$ and $q < 10^6$:

$$q = 2, \quad a_1 = 1.$$

Upper bound on the cardinality q

Proposition

Let $n \in \mathbb{Z}_{\geq 13}$. There exists a $q_n \in \mathbb{Z}$ such that if $-a_n = \lfloor 2\sqrt{q}^n \rfloor$ for some pair q, a_1, then $q \leq q_n$ or the pair q, a_1 is supersingular.

- Combined with the upper bound from linear forms in logarithms this implies that

Theorem

There exist only finitely many ordinary pairs q, a_1 such that $-a_n = \lfloor 2\sqrt{q}^n \rfloor$ for some $n \geq 13$.

Proof of proposition

Assume that the pair q, a_1 is ordinary and $-a_n = \lfloor 2\sqrt{q}^n \rfloor$.

- Recall that q not a square and n odd and
$$0 < |\beta^n + 1| < \frac{1}{\sqrt[4]{q}^n}.$$
- Observe that $\beta^n + 1 = \prod_{i=1}^n \left(\beta - \zeta_{2n}^{2i+1}\right)$.
- There is a $c_n > 0$ (depending only on n) and a m such that
$$|\beta^n + 1| \geq c_n |\beta - \zeta_{2n}^m| \geq c_n \left| \frac{a_1}{2\sqrt{q}} - \cos\left(\frac{m\pi}{n}\right) \right|.$$
- The Subspace Theorem implies: For all $\varepsilon > 0$ there exists a $c_0' > 0$ depending on $\cos\left(\frac{m\pi}{n}\right)$ and ε such that
$$\left| \frac{a_1}{2\sqrt{q}} - \cos\left(\frac{m\pi}{n}\right) \right| \geq \frac{c_0'}{(4q)^{3+\varepsilon}}.$$
- Take $\varepsilon = \frac{1}{8}$. Then $c < q^{3+\varepsilon-\frac{n}{4}}$ for some $c > 0$.

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Maximal over cubic extensions

Theorem

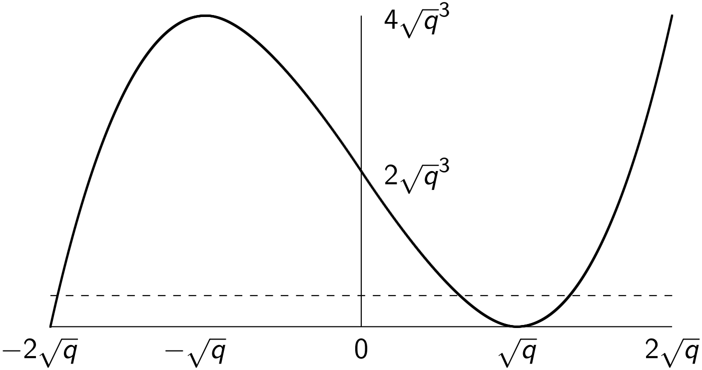
For infinitely many primes $q = p$ there exists an $a_1 \in \mathbb{Z}$ (with $|a_1| \leq 2\sqrt{q}$) such that $-a_3 = \left\lfloor 2\sqrt{q}^3 \right\rfloor$.

- Such a pair q, a_1 is ordinary.
- Using $a_{n+1} = a_1 a_n - q a_{n-1}$ and $a_0 = 2$

$$a_3 = a_1^3 - 3qa_1.$$

- Hence

$$-a_3 = \left\lfloor 2\sqrt{q}^3 \right\rfloor \quad \Longleftrightarrow \quad 0 \leq a_1^3 - 3qa_1 + 2\sqrt{q}^3 < 1.$$



Proposition (Soomro)

If $q = a_1^2 + b$ with integers a_1, b such that $a_1 \geq 2$ and $|b| \leq \sqrt{a_1}$, then $-a_3 = \lfloor 2\sqrt{q}^3 \rfloor$.

Proof of theorem

- Consider
$$S_1 = \left\{(a,b) \in \mathbb{Z}^2 : p = a^2 + b \text{ prime}, 0 < a, |b| \le \sqrt{a}\right\}$$
and define $S_2 = \{(a,b) \in S_1 : b \text{ square}\}$, which corresponds to
$$S_3 = \left\{(a,c) \in \mathbb{Z}^2 : p = a^2 + c^2 \text{ prime}, 0 < a, 0 \le c \le \sqrt[4]{a}\right\}.$$
- Define for $\theta > 0$
$$S_4(\theta) = \left\{(a,c) \in \mathbb{Z}^2 : p = a^2 + c^2 \text{ prime}, 0 < a, 0 \le c < p^{\theta}\right\}$$
and write $S_4(\theta) = S_5(\theta) \cup S_6(\theta)$ with
$$S_5(\theta) = \left\{(a,c) \in S_4(\theta) : a \ge p^{4\theta}\right\}$$
and
$$S_6(\theta) = \left\{(a,c) \in S_4(\theta) : a < p^{4\theta}\right\}.$$
- Observe that $S_5(\theta) \subset S_3$.
- If $\theta < \frac{1}{8}$, then $S_6(\theta)$ is finite, because $p = a^2 + c^2 < p^{8\theta} + p^{2\theta}$.
- The set $S_4(0.119)$ is infinite by Harman and Lewis (2001).