Errata
of
The arithmetic of maximal curves, the Hesse pencil and the Mestre curve

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Abstract

We provide a list of corrections to the author's PhD thesis [1]. The most recent version of this document can be found at `http://22gd7.nl/a.s.i.anema`.

Chapter 1: Elliptic curves maximal over finite extensions

- On page 3 in the 2nd line of the proof of Lemma 1.4 "$[\mathbb{Q}(\sqrt{q}, \beta), \mathbb{Q}]$" should read "$[\mathbb{Q}(\sqrt{q}, \beta) : \mathbb{Q}]$".
- Proposition 1.10 and Lemma 1.11 are wrong, because the proof of the lemma incorrectly assumes that -1 and β are multiplicatively independent. The statements are applied only in Subsection 1.2.3.

  Using [2, Théorème 3] instead of [2, Corollaire 1], Proposition 1.10 and Lemma 1.11 should read:

  Proposition. *Let q, a_1 be integers with $q \geq 2$ and $|a_1| \leq 2\sqrt{q}$. Suppose that N_q is the unique zero of*

  $$n \longmapsto \frac{n}{4}\log(q) - 8.87\left(10.98\pi + \frac{1}{2}\log(q)\right)(2\log(n) + 3.27)^2 - \log(2)$$

  larger than 8007*. If the pair q, a_1 is ordinary and $-a_n = \lfloor 2\sqrt{q}^n \rfloor$ for some n, then $n < N_q$.*

  Lemma. *Let β be an algebraic number of absolute value one. If β is not a root of unity, then*

  $$\log|\log(-\beta^n)| \geq -8.87(10.98\pi + dl)\max\left\{17, \frac{\sqrt{d}}{10}, d\log(n) - 0.88d + 5.03\right\}^2$$

  for all positive integers n, where l is an upper bound on the logarithmic height of β and $d = \frac{1}{2}[\mathbb{Q}(\beta) : \mathbb{Q}]$.

  Proof. Recall from the proof of Lemma 1.8 that

  $$\log(-\beta^n) = m\log(-1) + n\log(\beta)$$

  with m an odd integer such that $|m| \leq n$.

  Assume that m is negative. Let a and H be as in [2, Théorème 3]. Observe that

  $$20 \leq a \leq 10.98\pi + dl$$

  and using $20 \leq a$ and $|m| \leq n$ that

  $$H \leq \max\left\{17, \frac{\sqrt{d}}{10}, d\log(n) - 0.88d + 5.03\right\}.$$

Since $|\beta| = 1$ and β is not a root of unity, apply [2, Théorème 3] to obtain the desired lower bound.

Assume that m is positive. The logarithmic heights of β and $\bar{\beta}$ are equal and $\log(\bar{\beta}) = -\log(\beta)$. Replace β by $\bar{\beta}$ and apply [2, Théorème 3] to

$$\log(-\beta^n) = m\log(-1) - n\log(\bar{\beta}).$$

This gives the same lower bound as before. □

Proof of Proposition 1.10. Suppose that the pair q, a_1 is ordinary. If $-a_n = \lfloor 2\sqrt{q}^n \rfloor$ for some integer $n \geq 4$, then q is not a square by Proposition 1.6 and the minimal polynomial of β over $\mathbb{Z}$ has degree 4 and divides

$$qX^4 + \left(2q - a_1^2\right)X^2 + q$$

by Lemma 1.9, so that $[\mathbb{Q}(\beta) : \mathbb{Q}] = 4$. Since $|\beta| = 1$ and β is not a root of unity, β, $\bar{\beta}$, $-\beta$ and $-\bar{\beta}$ are the distinct roots of this polynomial so that the logarithmic height of β is at most $\frac{1}{4}\log(q)$. (The corrected) Lemma 1.11 gives

$$\log|\log(-\beta^n)| \geq -8.87\left(10.98\pi + \frac{1}{2}\log(q)\right)\max\{17, 2\log(n) + 3.27\}^2.$$

Moreover

$$|\log(-\beta^n)| < \frac{1}{\sqrt[4]{q}^n - 1} \leq 2\frac{1}{\sqrt[4]{q}^n}$$

by the proof of Proposition 1.7 and $\sqrt[4]{q}^n \geq 2$ for $q \geq 2$ and $n \geq 4$. Let n_0 be such that $17 = 2\log(n_0) + 3.27$, that is $n_0 = e^{6.865} \approx 958.1$. Define the function $f : \mathbb{R}_{\geq 1} \times \mathbb{R}_{\geq 1} \to \mathbb{R}$ as

$$(q, n) \longmapsto \frac{n}{4}\log(q) - 8.87\left(10.98\pi + \frac{1}{2}\log(q)\right)(2\log(n) + 3.27)^2 - \log(2).$$

Hence if $n \geq n_0$, then the lower and upper bounds imply $f(q, n) < 0$.

Observe that $f(1, n) < 0$ for all $n \geq 1$. The function $\mathbb{R}_{\geq 1} \to \mathbb{R}$

$$n \longmapsto q \cdot \frac{\partial f}{\partial q}(q, n) = \frac{n}{4} - \frac{8.87}{2}(2\log(n) + 3.27)^2$$

is independent of q, strictly convex, has a unique minimum $1243 < n_1 < 1244$ and has a unique zero $8007 < n_2 < 8008$ such that $n_2 > n_1$. Therefore $\frac{\partial f}{\partial q}(q, \lfloor n_2 \rfloor) < 0$ for all $q \geq 1$, and as a result $f(q, \lfloor n_2 \rfloor) < 0$ for all $q \geq 1$. On the other hand $n \mapsto f(q, n)$ is also strictly convex, because (for $q, n \geq 1$)

$$\frac{\partial^2 f}{\partial n^2}(q, n) = 35.48\left(10.98\pi + \frac{1}{2}\log(q)\right)\frac{2\log(n) + 1.27}{n^2} > 0.$$

Moreover if $q > 1$, then $f(q, n) > 0$ for sufficiently large n. Combined with $f(q, \lfloor n_2 \rfloor) < 0$ this shows that for all $q > 1$ there exists a unique $N_q > \lfloor n_2 \rfloor$ such that $f(q, N_q) = 0$.

Hence if $-a_n = \lfloor 2\sqrt{q}^n \rfloor$ for some positive integer $n \geq N_q$, then $n > \lfloor n_2 \rfloor > n_0$ and so $f(q, n) < 0$, which contradicts $f(q, n) \geq 0$ for all $n \geq N_q$. This proves the proposition. □

The results in Subsection 1.2.3 still hold using the upper bound in the above proposition.

- On page 8 in the 5th line "$[\mathbb{Q}(\beta) : \mathbb{Q}] = 2|\mathbb{R}(\beta) : \mathbb{R}|$" should read "$[\mathbb{Q}(\beta) : \mathbb{Q}] = 2[\mathbb{R}(\beta) : \mathbb{R}]$".
- On page 16 in the sentence above Proposition 1.18 "$a_1 = \lfloor \sqrt{q} \rfloor$" should read "$a_1 = \lceil \sqrt{q} \rceil$".
- On page 17 in the 5th line again "$a_1 = \lfloor \sqrt{q} \rfloor$" should read "$a_1 = \lceil \sqrt{q} \rceil$".

Chapter 3: Hesse pencil and Galois action on 3-torsion

- On page 34 in the last line of the last equation of the proof of Proposition 3.8 "$\alpha_1\lambda_1 u_1 + \alpha_2\lambda_2 u_2 + \alpha_3\lambda_3 u_3$," should read "$\alpha_1\lambda_1 v_1 + \alpha_2\lambda_2 v_2 + \alpha_3\lambda_3 v_3$,".

Chapter 4: Jacobian variety of the Mestre curve

- On page 43 in the 9th line of Section 4.3 "isomorphic $\mathbb{P}^1$" should read "isomorphic to $\mathbb{P}^1$".

Chapter 5: Faltings method

- On page 55 the 4th line should read "$G \longrightarrow \delta(G) \longrightarrow \delta(G)/\delta(N)^{p^e}$".
- On page 56 in the first line of Proposition 5.8 "$N \subset G$ be an open subgroup" should read "$N \subset G$ be an open normal subgroup".

Chapter 6: Galois extensions with exponent four group

- On page 75 in the -7th line "$q = v\sqrt[n]{3}^2$" should read "$\sqrt[n]{q} = v\sqrt[n]{3}^2$".

Bibliography

- On page 97 in the 64th entry the author is "N.P. Smart".

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- On page 99 the page number should not be there.

References

[1] A.S.I. Anema. *The arithmetic of maximal curves, the Hesse pencil and the Mestre curve.* PhD thesis, Rijksuniversiteit Groningen, 2016.

[2] M. Laurent, M. Mignotte, and Y. Nesterenko. Formes linéaires en deux logarithmes et déterminants d'interpolation. *Journal of Number Theory*, 55(2):285–321, 1995.