Rijksuniversiteit Groningen

3-Lie algebras

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Chapter 1

Introduction

For a long time Lie algebras are known and used in both mathematics and physics. They have been studied intensively in order to make a classification of Lie algebras. In physics for example they play an import role in quantum mechanics.

In essence a Lie algebra is a vector space $\mathcal{V}$ with a bilinear map $\mathcal{V}^2 \to \mathcal{V},\ (x, y) \mapsto [x, y]$. This map must be skew-symmetric and satisfies the so called Jacobi identity

$$[[x, y], z] + [[z, x], y] + [[y, z], x] = 0$$

for all $x, y, z \in \mathcal{V}$. The bilinear map can be regarded as a kind of multiplication. Therefore subalgebras, ideals and other concepts known from ring theory can also be defined for Lie algebras.

In 1985 V.T. Filippov [6] generalized Lie algebras to what is called n-Lie algebras or Filippov algebras. The bilinear map is replaced with a n-linear map $\mathcal{V}^n \to \mathcal{V},\ (x_1, \ldots, x_n) \mapsto [x_1, \ldots, x_n]$. This map must also be skew-symmetric and satisfies a generalization of the Jacobi identity such that it coincides with the definition of a Lie algebra if $n = 2$. Concepts known from Lie algebras are generalized to n-Lie algebras. In 1987 S.M. Kasymov [12] introduced additional concepts known from Lie algebras and for example showed that the definition of nilpotent can be generalized in several ways.

In physics applications of n-Lie algebras are found in string theory, or more specifically in M-theory. A model of multiple M2-branes is given in [2]. In this model a skew-symmetric 3-linear map is needed which satisfies an identity called the fundamental identity. Actually this identity is the same as the generalized Jacobi identity. This fact is recognized in [11] where new ways to construct 3-Lie algebras are studied. The n-Lie algebras also play a central role in a generalization of reduced super Yang-Mills theory [9].

The study of n-Lie algebras is related to the study of Nambu dynamics. In 1973 Y. Nambu [13] generalized Hamiltonian dynamics such that the dynamics is determined by multiple Hamiltonians. The definitions of Nambu-Poisson manifolds and Nambu-Lie algebras of order n is given by L. Takhtajan in 1994 [16]. In fact the definition of a Nambu-Lie algebra of order n coincides with the definition of a n-Lie algebra, as is noticed in [4, 11, 14].

A generalization of 3-Lie algebras appeared in [5]. The 3-linear map of a generalized 3-Lie algebra need not be skew-symmetric, but still satisfies the Jacobi identity.

Chapter 2 is about the definition of a n-Lie algebra and some general statements about n-Lie algebras. In chapter 3 we focus on finding 3-Lie algebras and show that quaternions naturally give a 3-Lie algebra. Furthermore in appendix A we develop some theorems concerning congruent matrices.

Chapter 2

n-Lie algebras

2.1 Definitions

In the articles [6, 12][1] most standard notations from the theory of Lie algebras are extended to so called *n-Lie algebras*, and what it means for such an algebra to be *k-solvable*, *k-nilpotent* or *simple*. There are no explicit definitions of *subalgebra*, *homomorphism* and *isomorphism*, but it is straightforward to define them as we will see below.

We begin with the definition of a *n-Lie algebra*. As we will see it is a natural generalization of a Lie algebra.

Definition 2.1. If $\mathcal{L}$ is a vector space with a n-linear map $[\cdot, \dots, \cdot] : \mathcal{L}^n \to \mathcal{L}$ such that the map is skew-symmetric

$$\left[x_{\sigma(1)}, \dots, x_{\sigma(n)}\right] = \epsilon(\sigma)\left[x_1, \dots, x_n\right] \tag{2.1}$$

and satisfies the *Jacobi identity*

$$\left[\left[x_1, \dots, x_n\right], y_2, \dots, y_n\right] = \sum_{i=1}^{n} \left[x_1, \dots, \left[x_i, y_2, \dots, y_n\right], \dots x_n\right] \tag{2.2}$$

for all elements $x_i, y_j \in \mathcal{L}$ and permutations $\sigma \in S_n$ with sign $\epsilon(\sigma) \in \{\pm 1\}$, then $\mathcal{L}$ is called a *n-Lie algebra*.

Next we give the definitions of a *subalgebra*, *abelian* n-Lie algebra and an *ideal* in a n-Lie algebra. Also these are very similar to the case of Lie algebras.

Definition 2.2. Let $\mathcal{L}$ be a n-Lie algebra. If $\mathcal{K}$ is a subspace of $\mathcal{L}$ such that $[k_1, \dots, k_n] \in \mathcal{K}$ for all $k_1, \dots, k_n \in \mathcal{K}$, then $\mathcal{K}$ is called a *n-Lie subalgebra*.

Definition 2.3. Let $\mathcal{L}$ be a n-Lie algebra. If $[x_1, \dots, x_n] = 0$ for all $x_1, \dots, x_n \in \mathcal{L}$, then the n-Lie algebra $\mathcal{L}$ is called *abelian*.

Definition 2.4. Let $\mathcal{L}$ be a n-Lie algebra. If $\mathcal{K}$ is a subspace of $\mathcal{L}$ such that $[k, x_2, \dots, x_n] \in \mathcal{K}$ for all $k \in \mathcal{K}$ and $x_2, \dots, x_n \in \mathcal{L}$, then $\mathcal{K}$ is called a *ideal*.

If the ideal $\mathcal{K}$ is unequal to $\{0\}$ and unequal to $\mathcal{L}$, then the ideal $\mathcal{K}$ is called *proper*.

[1] In these articles a slightly different notation is used. Suppose A is a linear map to act upon an element x. In the articles this is denoted as xA, whereas we use the notation Ax or $A(x)$.

Definition 2.5. Let $\mathcal{L}$ be a n-Lie algebra. If $\mathcal{L}$ is not abelian and has no proper ideals, then $\mathcal{L}$ is called *simple*.

The definitions of *homomorphism* and *isomorphism* can be generalized without problems from the definitions for Lie algebras in [3].

Definition 2.6. Suppose $\mathcal{L}_1$ and $\mathcal{L}_2$ are n-Lie algebras. A map $T : \mathcal{L}_1 \to \mathcal{L}_2$ is called a *homomorphism* if T is linear and satisfies

$$T\left([x_1, \ldots, x_n]_1\right) = \left[T\left(x_1\right), \ldots, T\left(x_n\right)\right]_2 \tag{2.3}$$

for all $x_i \in \mathcal{L}_1$.

Definition 2.7. Let $\mathcal{L}_1$ and $\mathcal{L}_2$ be n-Lie algebras. A bijective homomorphism $T : \mathcal{L}_1 \to \mathcal{L}_2$ is called an *isomorphism*. If there exists such an isomorphism T, then $\mathcal{L}_1$ and $\mathcal{L}_2$ are called *isomorphic*.

If $\mathcal{I}$ is an ideal of a n-Lie algebra $\mathcal{L}$, then we can define a new n-Lie algebra on the quotient space $\mathcal{L}/\mathcal{I}$ with the n-linear map

$$[\bar{v}_1, \ldots, \bar{v}_n]_{\mathcal{L}/\mathcal{I}} = [v_1, \ldots, v_n]_{\mathcal{L}} + \mathcal{I} \tag{2.4}$$

for all $\bar{v}_i \in \mathcal{L}/\mathcal{I}$ [6].

Theorem 2.8. *If $\mathcal{I}$ is an ideal of a n-Lie algebra $\mathcal{L}$, then the quotient space $\mathcal{L}/\mathcal{I}$ is also a n-Lie algebra and is called* n-Lie quotient algebra*.*

We know that a Lie algebra can be solvable or nilpotent. Similar a n-Lie algebra can also be solvable or nilpotent, but there are different generalizations possible [12]. Before we define *k-solvable* and *k-nilpotent*, a new notation is introduced in the following definition.

Definition 2.9. Let $\mathcal{L}$ be a n-Lie algebra. If $\mathcal{K}_1, \ldots, \mathcal{K}_n$ are subsets of $\mathcal{L}$, then define $[\mathcal{K}_1, \ldots, \mathcal{K}_n]$ to be the vector space

$$[\mathcal{K}_1, \ldots, \mathcal{K}_n] = \operatorname{span}\left\{[k_1, \ldots, k_n] : k_1 \in \mathcal{K}_1, \ldots, k_n \in \mathcal{K}_n\right\}. \tag{2.5}$$

Definition 2.10. Let $\mathcal{I}$ be an ideal of a n-Lie algebra $\mathcal{L}$ and $2 \le k \le n$. Define $\mathcal{I}_{(0,k)} = \mathcal{I}$ and $\mathcal{I}_{(r+1,k)} = \left[\mathcal{I}_{(r,k)}, \ldots, \mathcal{I}_{(r,k)}, \mathcal{L}, \ldots, \mathcal{L}\right]$ with k times $\mathcal{I}_{(r,k)}$ and $n-k$ times $\mathcal{L}$. If there exists a $r \ge 0$ such that $\mathcal{I}_{(r,k)} = 0$, then $\mathcal{I}$ is called *k-solvable*.

Definition 2.11. Let $\mathcal{I}$ be an ideal of a n-Lie algebra $\mathcal{L}$ and $2 \le k \le n$. Define $\mathcal{I}^{(1,k)} = \mathcal{I}$ and $\mathcal{I}^{(r+1,k)} = \left[\mathcal{I}^{(r,k)}, \mathcal{I}, \ldots, \mathcal{I}, \mathcal{L}, \ldots, \mathcal{L}\right]$ with $k-1$ times $\mathcal{I}$ and $n-k$ times $\mathcal{L}$. If there exists a $r \ge 1$ such that $\mathcal{I}^{(r,k)} = 0$, then $\mathcal{I}$ is called *k-nilpotent*.

2.2 General properties

In this section we give some examples of n-Lie algebras and some general properties of n-Lie algebras.

A vector space with a zero n-linear map is a trivial example of a abelian n-Lie algebra. From [6] we know an example of an infinite dimensional n-Lie algebra.

Example 2.12. Let $\mathcal{V} = \mathbb{R}[x_1, \ldots, x_n]$ be the vector space of polynomials in n variables. The vector space $\mathcal{V}$ with the Jacobian as the n-linear map

$$[f_1, \ldots, f_n] := \det \begin{pmatrix} \frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial f_n}{\partial x_1} & \cdots & \frac{\partial f_n}{\partial x_n} \end{pmatrix} \tag{2.6}$$

for $f_1, \ldots, f_n \in \mathcal{V}$, is a n-Lie algebra. For the proof see [7].

Given two n-Lie algebras over the same field, we can create a new n-Lie algebra on the direct sum of vector space of both n-Lie algebras.

Theorem 2.13. *If $\mathcal{L}_1$ and $\mathcal{L}_2$ are n-Lie algebras over the same field, then the direct sum $\mathcal{L}_1 \oplus \mathcal{L}_2$ with the n-linear map*

$$[x_1 \oplus y_1, \ldots, x_n \oplus y_n]_\oplus = [x_1, \ldots, x_n]_1 \oplus [y_1, \ldots, y_n]_2 \tag{2.7}$$

is also a n-Lie algebra.

Suppose we have a n-Lie algebra with $n > 2$. We can construct a m-Lie algebra where $m = n - 1$, by keeping one element fixed in the n-linear map [6].

Theorem 2.14. *Let $\mathcal{L}$ be a n-Lie algebra with $n > 2$. If $\mathcal{V}$ is the vector space of $\mathcal{L}$ and $m = n - 1$, then for any $x \in \mathcal{V}$ the vector space $\mathcal{V}$ with the m-linear map defined as*

$$[y_1, \ldots, y_m]_x = [y_1, \ldots, y_m, x] \tag{2.8}$$

for all $y_1, \ldots, y_m \in \mathcal{V}$, is a m-Lie algebra.

Using the above theorem inductively, we can construct a Lie algebra from a n-Lie algebra. Another way to create a Lie algebra form a n-Lie algebra is given in the following theorem.

Theorem 2.15. *If $\mathcal{L}$ is a n-Lie algebra, then the vector space of all derivations on $\mathcal{L}$, that is, all linear maps $D : \mathcal{L} \to \mathcal{L}$ such that*

$$D([x_1, \ldots, x_n]) = \sum_{i=1}^{n} [x_1, \ldots, D(x_i), \ldots, x_n]$$

for all $x_1, \ldots, x_n \in \mathcal{L}$, is a Lie algebra with the bilinear map given by

$$\mathcal{L} \times \mathcal{L} \to \mathcal{L}, \quad (D_1, D_2) \mapsto [D_1, D_2] = D_1 D_2 - D_2 D_1$$

for derivations D_1, D_2.

Chapter 3

3-Lie algebras

In this chapter we concentrate on finding n-Lie algebras in the case n is equal to three. We assume the characteristic of the field to be unequal to two. We take two approaches to finding all 3-Lie algebras of dimension lower than or equal to four. The former approach is quite naive and the latter is more formal and structured.

All n-Lie algebras of dimensions lower than or equal to $n+1$ have been calculated in [6]. We give a proof for four-dimensional 3-Lie algebras with the use of linear and multilinear algebra in the second approach.

We make a note first. It is easy to construct Lie algebras. Define on a vector space of square matrices the following bilinear map

$$[A, B] := AB - BA$$

for all matrices A, B in that vector spaces. This map satisfies the conditions of a Lie algebra as a straightforward computation shows. A reasonable generalization for a 3-linear map would be

$$[A, B, C] := ABC + BCA + CAB - BAC - ACB - CBA$$

for all matrices A, B, C in that vector spaces. This map is skew-symmetric, but unfortunately the map does not satisfy the Jacobi identity in general.

Example 3.1. Let $[\cdot, \cdot, \cdot]$ be as above. Choose six 3×3 matrices as follows

$$A_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad A_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad A_3 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$
$$A_4 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad A_5 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \qquad A_6 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

then $[A_1, [A_2, A_4, A_5], A_3] = -A_6$ and

$$[[A_1, A_2, A_3], A_4, A_5] = [[A_1, A_4, A_5], A_2, A_3] = [A_1, A_2, [A_3, A_4, A_5]] = O.$$

Insert these terms into the Jacobi identity, and we find that this identity is not satisfied for these matrices.

3.1 3-Lie algebras of dimension three and lower

Suppose $\mathcal{V}$ is a vector space of dimension m. A first approach to find all 3-Lie algebras on this vector space, is to find all skew-symmetric maps $[\cdot,\cdot,\cdot] : \mathcal{V}^3 \to \mathcal{V}$. By selecting the skew-symmetric maps which satisfy the Jacobi identity, one finds all possible 3-Lie algebras of dimension m.

Example 3.2. Let $\mathcal{V}$ be a vector space of dimension 0, then $\mathcal{V} = \{0\}$. The only possible map is $[0,0,0] = 0$. This map is skew-symmetric and satisfies the Jacobi identity.

Example 3.3. Let $\mathcal{V}$ be a vector space of dimension 1, then $\mathcal{V}$ has basis $\{e_1\}$. Since every element of $\mathcal{V}$ is equal to e_1 up to a scaler, we only need to define $[e_1, e_1, e_1]$. Skew-symmetry implies

$$[e_1, e_1, e_1] = -[e_1, e_1, e_1] \tag{3.1}$$

which can only hold if $[e_1, e_1, e_1] = 0$. Therefore $[x, y, z] = 0$ for all $x, y, z \in \mathcal{V}$. This map also satisfies the Jacobi identity.

The previous two examples show that the only possible 3-Lie algebras of dimension 0 and 1 are those with the map $[x, y, z] = 0$ for all $x, y, z \in \mathcal{V}$. This also holds for 3-Lie algebras of dimension 2, as can be proven in the same way as above.

Example 3.4. Let $\mathcal{V}$ be a vector space of dimension three with $\{e_1, e_2, e_3\}$ a basis of $\mathcal{V}$. If the 3-linear map is determined for all combinations of elements in the basis, then the map is also determined for all elements in $\mathcal{V}$. The skew-symmetry of the map implies

$$[e_i, e_j, e_k] = 0 \quad \text{if } i = j \text{ or } i = k \text{ or } j = k. \tag{3.2}$$

Therefore only $[e_1, e_2, e_3]$ can be chosen freely. The choice of $[e_1, e_2, e_3]$ can be split in two cases

- If $[e_1, e_2, e_3] = 0$, then $[x, y, z] = 0$ for all $x, y, z \in \mathcal{V}$.
- If $[e_1, e_2, e_3] \neq 0$, then there exists a $z = z_1e_1 + z_2e_2 + z_3e_3 \in \mathcal{V}$ unequal to zero such that $[e_1, e_2, e_3] = z$. Assume $z_1 \neq 0$, then $f_1 = z$, $f_2 = {z_1}^{-1}e_2$ and $f_3 = e_3$ also form a basis for $\mathcal{V}$ and $[f_1, f_2, f_3] = f_1$. The f_1,f_2 and f_3 can be chosen in a similar way, when $z_2 \neq 0$ or $z_3 \neq 0$.

Both choices of the map $[\cdot,\cdot,\cdot]$ satisfy the Jacobi identity. So there exist only two 3-Lie algebras of dimension three up to an isomorphism.

The previous example shows there are only two 3-Lie algebras of dimension three, and that any skew-symmetric map on a three dimension 3-Lie algebra satisfies the Jacobi identity. This is a special case of the following theorem.

Theorem 3.5. *Let $\mathcal{V}$ be some vector space. If $[\cdot,\cdot,\cdot] : \mathcal{V}^3 \to \mathcal{V}$ is a 3-linear skew-symmetric map, then for all $x, y, z \in \mathcal{V}$ the map satisfies*

$$[[x, y, z], x, y] = [[x, x, y], y, z] + [x, [y, x, y], z] + [x, y, [z, x, y]]\,. \tag{3.3}$$

Proof. The skew-symmetry of the map implies $[x, x, y] = 0$, $[y, x, y] = 0$ and $[z, x, y] = [x, y, z]$ for all $x, y, z \in \mathcal{V}$. These equalities imply

$$\begin{aligned}[][[x, y, z], x, y] &= [0, y, z] + [x, 0, z] + [[z, x, y], x, y] \\ &= [[x, x, y], y, z] + [x, [y, x, y], z] + [x, y, [z, x, y]]\end{aligned}$$

which is equal to equation (3.3). □

All 3-Lie algebras of dimension four can be found by a similar method as above, but there is an important difference. In this case there exist skew-symmetric maps which do not satisfy the Jacobi identity, as the following example illustrates.

Example 3.6. Let $\mathcal{V}$ be a vector space of dimension 4 with basis $\{e_1, e_2, e_3, e_4\}$. Fix a skew-symmetric map $[\cdot, \cdot, \cdot] : \mathcal{V}^3 \to \mathcal{V}$ by

$$\begin{aligned}[][e_1, e_2, e_3] &= e_1 \\ [e_1, e_2, e_4] &= 0 \\ [e_1, e_3, e_4] &= 0 \\ [e_2, e_3, e_4] &= e_2.\end{aligned}$$

This map does not satisfy the Jacobi identity, because

$$[[e_1, e_2, e_3], e_3, e_4] = 0$$

and

$$[[e_1, e_3, e_4], e_2, e_3] + [e_1, [e_2, e_3, e_4], e_3] + [e_1, e_2, [e_3, e_3, e_4]] = e_1$$

are not equal.

Example 3.7. Let $\mathcal{V}$ be a vector space with $\dim \mathcal{V} = 4$ and $\mathcal{W} = [\mathcal{V}, \mathcal{V}, \mathcal{V}]$. If $\dim \mathcal{W} = 0$, then $\mathcal{W} = \{0\}$. Therefore $[x, y, z] = 0$ for all $x, y, z \in \mathcal{V}$. This map is skew symmetric and satisfies the Jacobi identity.

Example 3.8. Let $\mathcal{V}$ be a vector space with $\dim \mathcal{V} = 4$ and $\mathcal{W} = [\mathcal{V}, \mathcal{V}, \mathcal{V}]$. If $\dim \mathcal{W} = 1$, then $\mathcal{W}$ has a basis $\{e_1\}$. There are two possible cases for a skew-symmetric map

- Assume $[e_1, x, y] = 0$ for all $x, y \in \mathcal{V}$. Extend the basis $\{e_1\}$ of $\mathcal{W}$ to a basis $\{e_1, e_2, e_3, e_4\}$ of $\mathcal{V}$. The assumption implies

  $$[e_1, e_2, e_3] = [e_1, e_2, e_4] = [e_1, e_3, e_4] = 0.$$

  Since $\dim \mathcal{W} = 1$ there must exist a nonzero $z = z_1 e_1 \in \mathcal{W}$ such that $[e_2, e_3, e_4] = z$. Choose a new basis $\{f_1, f_2, f_3, f_4\}$ of $\mathcal{V}$ with $f_1 = e_1$, $f_2 = e_2$, $f_3 = e_3$ and $f_4 = {z_1}^{-1} e_4$. The skew-symmetric map is fixed by

  $$[f_1, f_2, f_3] = [f_1, f_2, f_4] = [f_1, f_3, f_4] = 0$$

  and

  $$[f_2, f_3, f_4] = f_1.$$

- Assume there exist $e_2, e_3 \in \mathcal{V}$ such that $[e_1, e_2, e_3] \neq 0$. The skew-symmetry of the map implies e_1, e_2 and e_3 are linear independent. The basis $\{e_1\}$ of $\mathcal{W}$ can be extended to a basis $\{e_1, e_2, e_3, e_4\}$ of $\mathcal{V}$ for some $e_4 \in \mathcal{V}$. Since $[e_1, e_2, e_3] \neq 0$ and $\{e_1\}$ is a basis of $\mathcal{W}$, there exists $\alpha, \beta, \gamma \in \mathbb{F}$ and a nonzero $z = z_1 e_1 \in \mathcal{W}$ such that

$$\begin{aligned}
[e_1, e_2, e_3] &= z \\
[e_1, e_2, e_4] &= \alpha z \\
[e_1, e_3, e_4] &= \beta z \\
[e_2, e_3, e_4] &= \gamma z.
\end{aligned}$$

  Choose a new basis $\{f_1, f_2, f_3, f_4\}$ of $\mathcal{V}$ with $f_1 = z$, $f_2 = {z_1}^{-1} e_2$, $f_3 = e_3$ and $f_4 = e_4 - \alpha e_3 + \beta e_2 - \gamma e_1$. The skew-symmetric map is fixed by

$$[f_1, f_2, f_3] = f_1$$

  and

$$[f_1, f_2, f_4] = [f_1, f_3, f_4] = [f_2, f_3, f_4] = 0.$$

Both choices of the skew-symmetric map satisfy the Jacobi identity.

The previous two examples illustrate a method to find all 3-Lie algebras of dimension four. First find all 3-Lie algebras with $\dim [\mathcal{V}, \mathcal{V}, \mathcal{V}] = 0$, then find all 3-Lie algebras with $\dim [\mathcal{V}, \mathcal{V}, \mathcal{V}] = 1$ and so on.

The methods presented in this section can be used to find all 3-Lie algebras of dimensions up to four. The same methods can also be used for higher finite dimensions, but is not recommended. The number of skew-symmetric 3-linear maps increases as the dimension of the vector space increases, while only some skew-symmetric maps satisfy the Jacobi identity. This can result in a lot of useless calculations.

3.2 Four dimensional 3-Lie algebras

In the previous section we have seen a method to find all 3-Lie algebras of finite dimensions, but the section also showed this will be a lot of work and might be error prone. We need a more structured approach to search for 3-Lie algebras.

We give a short introduction to multilinear algebra. For a more complete introduction we redirect the reader to [10] or any other book on this subject.

Suppose $\mathcal{V}$ and $\mathcal{W}$ are vector spaces. Then there exists a vector space $\bigwedge^n \mathcal{V}$ with elements $x_1 \wedge \ldots \wedge x_n$ and an unique n-linear skew-symmetric map

$$\mathcal{V}^n \to \bigwedge^n \mathcal{V}, \qquad (x_1, \ldots, x_n) \mapsto x_1 \wedge \ldots \wedge x_n$$

such that for any skew-symmetric n-linear map $f : \mathcal{V}^n \to \mathcal{W}$ there exists a unique linear map $g : \bigwedge^n \mathcal{V} \to \mathcal{W}$ that satisfies $f(x_1, \ldots, x_n) = g(x_1 \wedge \ldots \wedge x_n)$. That is, the following diagram commutes

$$\begin{tikzcd}
\mathcal{V}^n \arrow[r, "f"] \arrow[d, "\wedge^n"'] & \mathcal{W} \\
\bigwedge^n \mathcal{V} \arrow[ur, "g"'] &
\end{tikzcd}$$

If $\mathcal{V}$ is a n dimensional vector space, then $\binom{n}{k}$ equals the dimension of $\bigwedge^k \mathcal{V}$.

Example 3.9. Let $T : \mathcal{V} \to \mathcal{W}$ be a linear map between vector space $\mathcal{V}$ and $\mathcal{W}$ and k some positive integer. We define a skew-symmetric k-linear map as

$$\mathcal{V}^k \to \bigwedge^k \mathcal{W}, \qquad (v_1, \ldots, v_k) \mapsto Tv_1 \wedge \ldots \wedge Tv_k$$

for all $v_1, \ldots, v_k \in \mathcal{V}$. This map induces a unique linear map

$$T^{\wedge^k} : \bigwedge^k \mathcal{V} \to \bigwedge^k \mathcal{W}, \qquad (v_1 \wedge \ldots \wedge v_k) \mapsto Tv_1 \wedge \ldots \wedge Tv_k$$

for all $v_1 \wedge \ldots \wedge v_k \in \bigwedge^k \mathcal{V}$.

A 3-Lie algebra is a vector space with a skew-symmetric 3-linear map. The 3-linear map can be decomposed into a universal skew-symmetric map and a linear map. The following theorem shows the Jacobi identity is a restriction on the linear map.

Theorem 3.10. *Let $\mathcal{V}$ be a vector space and $[\cdot,\cdot,\cdot] : \mathcal{V}^3 \to \mathcal{V}$ a 3-linear map. The vector space $\mathcal{V}$ with the 3-linear map is a 3-Lie algebra if and only if there exists a linear map $\rho : \bigwedge^3 \mathcal{V} \to \mathcal{V}$ such that $[v_1, v_2, v_3] = \rho(v_1 \wedge v_2 \wedge v_3)$ for all $v_1, v_2, v_3 \in \mathcal{V}$ and satisfies*

$$\begin{aligned} \rho(\rho(v_1 \wedge v_2 \wedge v_3) \wedge v_4 \wedge v_5) = {} & \rho(\rho(v_1 \wedge v_4 \wedge v_5) \wedge v_2 \wedge v_3) \\ & + \rho(v_1 \wedge \rho(v_2 \wedge v_4 \wedge v_5) \wedge v_3) \\ & + \rho(v_1 \wedge v_2 \wedge \rho(v_3 \wedge v_4 \wedge v_5)) \end{aligned} \tag{3.4}$$

for all $v_1, v_2, v_3, v_4, v_5 \in \mathcal{V}$.

The next theorem states that every 3-Lie algebra of dimension lower than three is trivial. This is the direct consequence of the dimension of $\bigwedge^3 \mathcal{V}$ which is zero in this case.

Theorem 3.11. *All 3-Lie algebras $\mathcal{L}$ of dimension lower than three over a field $\mathbb{F}$ of characteristic unequal to two are isomorphic to the vector space $\mathbb{F}^n$ with the map $[x, y, z] = 0$ for all $x, y, z \in \mathbb{F}^n$ with $n = \dim \mathcal{L}$.*

All 3-Lie algebras of dimension three are given in the following theorem, which is stated without proof. The result is the same as in example 3.4.

Theorem 3.12. *All three dimensional 3-Lie algebras over a field $\mathbb{F}$ of characteristic unequal to two are isomorphic to one of the following 3-Lie algebras*

- *The vector space $\mathbb{F}^3$ with the map $[x, y, z] = 0$ for all $x, y, z \in \mathbb{F}^3$.*
- *The vector space $\mathbb{F}^3$ with the skew-symmetric multilinear map fixed by $[e_1, e_2, e_3] = e_1$, where $\{e_1, e_2, e_3\}$ is the standard ordered basis of $\mathbb{F}^3$.*

The four dimensional 3-Lie algebras up to an isomorphism are given in the next theorem. Some of the 3-Lie algebras can still be isomorphic to each other as will be clear from the proof of the theorem and a corollary about the special case when the field $\mathbb{F}$ is algebraically closed.

Theorem 3.13. *All four dimensional 3-Lie algebras over a field $\mathbb{F}$ of characteristic unequal to two are isomorphic to one of the following 3-Lie algebras*

- *The vector space $\mathbb{F}^4$ with the skew-symmetric 3-linear map given by*

$$\begin{aligned}[e_1, e_2, e_3] &= \alpha_4 e_4 \\ [e_1, e_2, e_4] &= \alpha_3 e_3 \\ [e_1, e_3, e_4] &= \alpha_2 e_2 \\ [e_2, e_3, e_4] &= \alpha_1 e_1\end{aligned}$$

  where $\{e_1, e_2, e_3, e_4\}$ is the standard ordered basis of $\mathbb{F}^4$ and $\alpha_1, \alpha_2, \alpha_3$ and α_4 are elements from $\mathbb{F}$ such that

$$\alpha_1 = 0 \quad \Rightarrow \quad \alpha_2 = 0 \quad \Rightarrow \quad \alpha_3 = 0 \quad \Rightarrow \quad \alpha_4 = 0.$$

- *The vector space $\mathbb{F}^4$ with the skew-symmetric 3-linear map given by*

$$\begin{aligned}[e_1, e_2, e_3] &= 0 \\ [e_1, e_2, e_4] &= 0 \\ [e_1, e_3, e_4] &= e_1 + \alpha_2 e_2 \\ [e_2, e_3, e_4] &= e_2 - \alpha_1 e_1\end{aligned}$$

  where $\{e_1, e_2, e_3, e_4\}$ is the standard ordered basis of $\mathbb{F}^4$ and α_1 and α_2 are elements from $\mathbb{F}$ such that $\alpha_1\alpha_2 + 1 \neq 0$ and

$$\alpha_1 = 0 \quad \Rightarrow \quad \alpha_2 = 0.$$

- *The vector space $\mathbb{F}^4$ with the skew-symmetric 3-linear map given by*

$$\begin{aligned}[e_1, e_2, e_3] &= e_1 \\ [e_1, e_2, e_4] &= 0 \\ [e_1, e_3, e_4] &= 0 \\ [e_2, e_3, e_4] &= 0\end{aligned}$$

  where $\{e_1, e_2, e_3, e_4\}$ is the standard ordered basis of $\mathbb{F}^4$.

The road to the proof of this theorem is as follows. We know that for each 3-Lie algebra there exists a linear map ρ such that $[x, y, z] = \rho(x \wedge y \wedge z)$. First we translate the condition of isomorphic 3-Lie algebras into a relation between their linear maps ρ. After this we give a set of equations in terms of the matrix representation of ρ that are equivalent to the Jacobi identity. Finally the proof of the theorem is presented.

Suppose $L : \mathcal{V} \to \mathcal{W}$ is a linear map between vector spaces $\mathcal{V}$ and $\mathcal{W}$. This map leads to a linear map $T^* : \mathcal{W}^* \to \mathcal{V}^*$ with

$$T^*(\omega)(v) := \omega(Tv)$$

for all $v \in \mathcal{V}$ and $\omega \in \mathcal{W}^*$.

From this point on until the proof of the theorem, we will assume that all 3-Lie algebras of dimension four are defined on the same vector space $\mathcal{V} = \mathbb{F}^4$ with an ordered basis $\{e_1, e_2, e_3, e_4\}$. Let $\{e^1, e^2, e^3, e^4\}$ be the basis of the dual vector space $\mathcal{V}^*$ such that $e^i(e_j) = \delta^i_j$.

The vector spaces $\bigwedge^3 \mathcal{V}$ and $\mathcal{V}^*$ both have dimension four. This allows us to choose an isomorphism of vector spaces between them.

Lemma 3.14. *Let $\mathcal{V}$ and $\mathcal{V}^*$ be vector spaces with bases as above. The linear map $\phi : \bigwedge^3 \mathcal{V} \to \mathcal{V}^*$ defined as*

$$\phi\left(v_1 \wedge v_2 \wedge v_3\right)(v) := \langle v_1 \wedge v_2 \wedge v_3 \wedge v, e_1 \wedge e_2 \wedge e_3 \wedge e_4 \rangle$$

for all $v_1 \wedge v_2 \wedge v_3 \in \bigwedge^3 \mathcal{V}$ and $v \in \mathcal{V}$ is an isomorphism of vector spaces, where $\langle \cdot, \cdot \rangle$ is an inner product on $\bigwedge^4 \mathcal{V}$ such that

$$\langle e_1 \wedge e_2 \wedge e_3 \wedge e_4, e_1 \wedge e_2 \wedge e_3 \wedge e_4 \rangle = 1.$$

So far we saw that a 3-Lie algebra on a vector space $\mathcal{V}$ corresponds to a linear map $\rho : \bigwedge^3 \mathcal{V} \to \mathcal{V}$. Using the above lemma we see that the 3-Lie algebra corresponds to a linear map $\tau : \mathcal{V}^* \to \mathcal{V}$, that is, $\tau := \rho \circ \phi^{-1}$.

The following lemma translates an isomorphism of 3-Lie algebras into a simple relation between the linear maps τ for each 3-Lie algebra and a linear map T.

Lemma 3.15. *Suppose $\mathcal{V}$ with $[\cdot,\cdot,\cdot]_1$ and $\mathcal{V}$ with $[\cdot,\cdot,\cdot]_2$ are two 3-Lie algebras of dimension four. Let $T : \mathcal{V} \to \mathcal{V}$ be a linear map. The map T is an isomorphism if and only if the map T is bijective and*

$$T \circ \tau_1 \circ T^* = \det(T)\, \tau_2$$

where $\tau_i : \mathcal{V}^* \to \mathcal{V}$ are the maps corresponding to $[\cdot,\cdot,\cdot]_i$.

Proof. Let $\mathcal{V}$ with $[\cdot,\cdot,\cdot]_1$ and $\mathcal{V}$ with $[\cdot,\cdot,\cdot]_2$ be two 3-Lie algebras and let $T : \mathcal{V} \to \mathcal{V}$ be a linear map. Then there exists linear maps $\rho_i : \bigwedge^3 \mathcal{V} \to \mathcal{V}$ such that

$$[v_1, v_2, v_3]_i = \rho_i\left(v_1 \wedge v_2 \wedge v_3\right)$$

for all $v_1, v_2, v_3 \in \mathcal{V}$. Furthermore there exists linear maps $\tau_i : \mathcal{V}^* \to \mathcal{V}$ such that the following diagram commutes.

$$\begin{tikzcd}
\mathcal{V}^* \arrow[dr, "\tau_1"'] & \bigwedge^3 \mathcal{V} \arrow[l, "\phi"'] \arrow[r, "T^{\wedge^3}"] \arrow[d, "\rho_1"] & \bigwedge^3 \mathcal{V} \arrow[r, "\phi"] \arrow[d, "\rho_2"'] & \mathcal{V}^* \arrow[dl, "\tau_2"] \\
 & \mathcal{V} \arrow[r, "T"'] & \mathcal{V} &
\end{tikzcd}$$

The map T is an isomorphism if and only if T is bijective and

$$T\,[v_1, v_2, v_3]_1 = [Tv_1, Tv_2, Tv_3]_2$$

for all $v_1, v_2, v_3 \in \mathcal{V}$. This equation is equivalent to

$$T \circ \tau_1 \circ \phi = \tau_2 \circ \phi \circ T^{\wedge^3}.$$

Define a linear map $S : \mathcal{V} \to \mathcal{V}$ such that $S^* := \phi \circ T^{\wedge^3} \circ \phi^{-1} \circ T^*$. Since the maps ϕ and T are bijective, then the above equation is equivalent to

$$T \circ \tau_1 \circ T^* = \tau_2 \circ S^*. \tag{3.5}$$

There exists an invertible linear map $S' : \mathcal{V} \to \mathcal{V}$ such that $S'^* \circ T^* = T^* \circ S^*$, because the maps T and S are invertible. Combining S'^* with the definition of S^* gives

$$S'^* \circ \phi = T^* \circ \phi \circ T^{\wedge^3}.$$

Writing out both sides of the equation results into

$$v_1 \wedge v_2 \wedge v_3 \wedge S'v = Tv_1 \wedge Tv_2 \wedge Tv_3 \wedge Tv$$

for all $v_1 \wedge v_2 \wedge v_3 \in \bigwedge^3 \mathcal{V}$ and $v \in \mathcal{V}$. But this equation is satisfied if and only if $S' = \det(T)\, I$. This map scales any vector with constant $\det(T)$, that is, also the maps S'^* and S^* scale any vector with the same constant. Thus equation (3.5) is equivalent to

$$T \circ \tau_1 \circ T^* = \det(T)\, \tau_2.$$

This completes the proof of the lemma. □

When we choose some basis of the vector space $\mathcal{V}$, then in matrix representation the relation which an isomorphism must satisfy is

$$Q\,[\tau_1]\,Q^T = \det(Q)\,[\tau_2]$$

for the matrix representation Q of the isomorphism. This equation looks like the relation between congruent matrices. This similarity lies at the heart of the next lemma.

Lemma 3.16. *If A is a 4×4 matrix over a field $\mathbb{F}$ of characteristic unequal to two, then there exists an invertible 4×4 matrix Q such that*

$$QAQ^T = \det(Q) \begin{pmatrix} \alpha_1 & \beta_1 & \gamma_1 & \gamma_2 \\ -\beta_1 & \alpha_2 & \gamma_3 & \gamma_4 \\ \gamma_1 & \gamma_3 & \alpha_3 & \beta_2 \\ \gamma_2 & \gamma_4 & -\beta_2 & \alpha_4 \end{pmatrix}$$

with scalars $\alpha_i, \beta_i, \gamma_i \in \mathbb{F}$. The scalars are constrained by

$$\begin{aligned} \beta_1 = 0 \quad &\Rightarrow \quad \beta_2 = 0 \\ \alpha_1 = 0 \quad &\Rightarrow \quad \alpha_2 = 0 \\ \alpha_3 = 0 \quad &\Rightarrow \quad \alpha_4 = 0 \end{aligned}$$

and if $\beta_1 = 0$, then $\gamma_1 = \gamma_2 = \gamma_3 = \gamma_4 = 0$ and

$$\alpha_2 = 0 \quad \Rightarrow \quad \alpha_3 = 0.$$

Proof. This lemma is essentially a special case of theorem A.4. That theorem states that a matrix A is congruent to a matrix B of a special shape, that is, there exists an invertible 4×4 matrix P such that $P^T AP = B$. Define $Q := P^T$ and $C := \det(Q)^{-1} B$.

The constraints on the scalars are due to the freedom to switch diagonal elements in specific parts in the matrix, and due to the special cases associated to the rank of the skew-symmetric part of C. □

We know that the Jacobi identity is a restriction on the linear map $\tau : \mathcal{V}^* \to \mathcal{V}$, or equivalent a restriction on its matrix representation. This is precisely the topic of the following lemma.

Lemma 3.17. *Let $\mathcal{V}$ be a four dimensional vector space with an ordered basis as described above. Suppose $[\cdot,\cdot,\cdot] : \mathcal{V}^3 \to \mathcal{V}$ is a 3-linear map and $\tau : \mathcal{V}^* \to \mathcal{V}$ a linear map such that $[x, y, z] = \tau \circ \phi (x \wedge y \wedge z)$ for all $x, y, z \in \mathcal{V}$. The matrix representation of τ is given by*

$$[\tau] = \begin{pmatrix} \tau_{11} & \tau_{12} & \tau_{13} & \tau_{14} \\ \tau_{21} & \tau_{22} & \tau_{23} & \tau_{24} \\ \tau_{31} & \tau_{32} & \tau_{33} & \tau_{34} \\ \tau_{41} & \tau_{42} & \tau_{43} & \tau_{44} \end{pmatrix}.$$

The vector space $\mathcal{V}$ with the 3-linear map $[\cdot,\cdot,\cdot]$ is a 3-Lie algebra if and only if τ satisfies the following set of equations

$$\begin{aligned}
(\tau_{34} - \tau_{43})\,\tau_{i2} + (\tau_{42} - \tau_{24})\,\tau_{i3} + (\tau_{23} - \tau_{32})\,\tau_{i4} = 0 \\
(\tau_{43} - \tau_{34})\,\tau_{i1} + (\tau_{14} - \tau_{41})\,\tau_{i3} + (\tau_{31} - \tau_{13})\,\tau_{i4} = 0 \\
(\tau_{24} - \tau_{42})\,\tau_{i1} + (\tau_{41} - \tau_{14})\,\tau_{i2} + (\tau_{12} - \tau_{21})\,\tau_{i4} = 0 \\
(\tau_{23} - \tau_{32})\,\tau_{i1} + (\tau_{31} - \tau_{13})\,\tau_{i2} + (\tau_{12} - \tau_{21})\,\tau_{i3} = 0
\end{aligned}$$

for $i = 1, 2, 3, 4$.

Proof. The vector space $\mathcal{V}$ with the skew-symmetric 3-linear map $[\cdot,\cdot,\cdot]$ is a 3-Lie algebra if and only if $[\cdot,\cdot,\cdot]$ satisfies the Jacobi identity

$$\begin{aligned}
0 = & \left[[v_1, v_2, v_3], v_4, v_5\right] - \left[[v_1, v_4, v_5], v_2, v_3\right] - \\
& \left[v_1, [v_2, v_4, v_5], v_3\right] - \left[v_1, v_2, [v_3, v_4, v_5]\right]
\end{aligned}$$

for all $v_1, v_2, v_3, v_4, v_5 \in \mathcal{V}$. It is necessary and sufficient to prove the Jacobi identity for all elements in the basis of $\mathcal{V}$.

As an example the Jacobi identity is written out for the vectors

$$(v_1, v_2, v_3, v_4, v_5) = (e_1, e_2, e_3, e_4, e_1).$$

Start by expressing $[e_i, e_j, e_k]$ in terms of $\tau_{i'j'}$

$$\begin{aligned}
[e_1, e_2, e_3] &= +\tau_{14}e_1 + \tau_{24}e_2 + \tau_{34}e_3 + \tau_{44}e_4 \\
[e_1, e_2, e_4] &= -\tau_{13}e_1 - \tau_{23}e_2 - \tau_{33}e_3 - \tau_{43}e_4 \\
[e_1, e_3, e_4] &= +\tau_{12}e_1 + \tau_{22}e_2 + \tau_{32}e_3 + \tau_{42}e_4 \\
[e_2, e_3, e_4] &= -\tau_{11}e_1 - \tau_{21}e_2 - \tau_{31}e_3 - \tau_{41}e_4.
\end{aligned}$$

Expand each term in the Jacobi identity in terms of τ_{ij}

$$\begin{aligned}
\left[[e_1, e_2, e_3], e_4, e_1\right] = & \left(-\tau_{24}\tau_{13} + \tau_{34}\tau_{12}\right) e_1 + \left(-\tau_{24}\tau_{23} + \tau_{34}\tau_{22}\right) e_2 + \\
& \left(-\tau_{24}\tau_{33} + \tau_{34}\tau_{32}\right) e_3 + \left(-\tau_{24}\tau_{43} + \tau_{34}\tau_{42}\right) e_4 \\
\left[[e_1, e_4, e_1], e_2, e_3\right] = & \, 0 \\
\left[e_1, [e_2, e_4, e_1], e_3\right] = & \left(-\tau_{23}\tau_{14} + \tau_{43}\tau_{12}\right) e_1 + \left(-\tau_{23}\tau_{24} + \tau_{43}\tau_{22}\right) e_2 + \\
& \left(-\tau_{23}\tau_{34} + \tau_{43}\tau_{32}\right) e_3 + \left(-\tau_{23}\tau_{44} + \tau_{43}\tau_{42}\right) e_4 \\
\left[e_1, e_2, [e_3, e_4, e_1]\right] = & \left(+\tau_{32}\tau_{14} - \tau_{42}\tau_{13}\right) e_1 + \left(+\tau_{32}\tau_{24} - \tau_{42}\tau_{23}\right) e_2 + \\
& \left(+\tau_{32}\tau_{34} - \tau_{42}\tau_{33}\right) e_3 + \left(+\tau_{32}\tau_{44} - \tau_{42}\tau_{43}\right) e_4.
\end{aligned}$$

Insert these terms into the Jacobi identity. This gives an equivalent equation in terms of τ_{ij} for the vectors $v_1, \dots, v_5$

$$\begin{aligned}0 = &\left[(\tau_{34} - \tau_{43})\,\tau_{12} + (\tau_{42} - \tau_{24})\,\tau_{13} + (\tau_{23} - \tau_{32})\,\tau_{14}\right] e_1 + \\ &\left[(\tau_{34} - \tau_{43})\,\tau_{22} + (\tau_{42} - \tau_{24})\,\tau_{23} + (\tau_{23} - \tau_{32})\,\tau_{24}\right] e_2 + \\ &\left[(\tau_{34} - \tau_{43})\,\tau_{32} + (\tau_{42} - \tau_{24})\,\tau_{33} + (\tau_{23} - \tau_{32})\,\tau_{34}\right] e_3 + \\ &\left[(\tau_{34} - \tau_{43})\,\tau_{42} + (\tau_{42} - \tau_{24})\,\tau_{43} + (\tau_{23} - \tau_{32})\,\tau_{44}\right] e_4.\end{aligned}$$

This is a subset of the set of sixteen quadratic equations.

The remaining equations are derived by choosing a different combination of basis vectors. This proves the lemma. □

At this point we have split the 3-linear map $[\cdot,\cdot,\cdot] : \mathcal{V}^3 \to \mathcal{V}$ into a skew-symmetric part $\phi\,(\cdot \wedge \cdot \wedge \cdot) : \mathcal{V}^3 \to \mathcal{V}^*$ and a linear part $\tau : \mathcal{V}^* \to \mathcal{V}$. For any four dimensional 3-Lie algebra on $\mathcal{V}$ we can find an isomorphic 3-Lie algebra using lemmas 3.15 and 3.16 such that the linear map τ corresponding to the latter has a convenient matrix representation with respect to the quadratic equations given in lemma 3.17. Now we are ready to prove theorem 3.13.

Proof of theorem 3.13. Suppose $\mathcal{L}$ is a four dimensional 3-Lie algebra over a field $\mathbb{F}$ of characteristic unequal to two. Let $[\cdot,\cdot,\cdot]_{\mathcal{L}} : \mathcal{L}^3 \to \mathcal{L}$ be the skew-symmetric 3-linear map associated with the 3-Lie algebra $\mathcal{L}$. As a vector space $\mathcal{L}$ is isomorphic with $\mathcal{V} = \mathbb{F}^4$, that is, there exists a invertible linear map $T : \mathcal{L} \to \mathcal{V}$. Define $[\cdot,\cdot,\cdot]_{\mathcal{V}} : \mathcal{V}^3 \to \mathcal{V}$ as

$$[v_1, v_2, v_3]_{\mathcal{V}} := T^{-1}\,[Tv_1, Tv_2, Tv_3]_{\mathcal{L}}\,.$$

This is a skew-symmetric 3-linear map on $\mathcal{V}$ and satisfies the Jacobi identity, because of $[\cdot,\cdot,\cdot]_{\mathcal{L}}$. Furthermore, the 3-Lie algebra $\mathcal{L}$ is isomorphic to the vector space $\mathcal{V}$ with the 3-linear map $[\cdot,\cdot,\cdot]_{\mathcal{V}}$. This shows that any four dimensional 3-Lie algebra over a field $\mathbb{F}$ of characteristic unequal to two is isomorphic to some 3-Lie algebra defined on $\mathcal{V} = \mathbb{F}^4$.

Suppose $\mathcal{V}$ with $[\cdot,\cdot,\cdot]_1 : \mathcal{V}^3 \to \mathcal{V}$ is a four dimensional 3-Lie algebra. Then there exists a linear map $\tau_1 : \mathcal{V}^* \to \mathcal{V}$ such that

$$[v_1, v_2, v_3]_1 = \tau_1 \circ \phi\,(v_1 \wedge v_2 \wedge v_3)$$

for all $v_1, v_2, v_3 \in \mathcal{V}$. Lemmas 3.15 and 3.16 imply the existence of a linear map $\tau_2 : \mathcal{V}^* \to \mathcal{V}$ such that the $\mathcal{V}$ with τ_1 and $\mathcal{V}$ with τ_2 are isomorphic 3-Lie algebras and the matrix representation of τ_2 is

$$[\tau_2] = \begin{pmatrix} \alpha_1 & \beta_1 & \gamma_1 & \gamma_2 \\ -\beta_1 & \alpha_2 & \gamma_3 & \gamma_4 \\ \gamma_1 & \gamma_3 & \alpha_3 & \beta_2 \\ \gamma_2 & \gamma_4 & -\beta_2 & \alpha_4 \end{pmatrix}$$

with scalars $\alpha_i, \beta_i, \gamma_i \in \mathbb{F}$ that satisfy the constraints given in lemma 3.16. This shows that any 3-Lie algebra on $\mathcal{V}$ is isomorphic to a 3-Lie algebra with a $\tau : \mathcal{V}^* \to \mathcal{V}$ whose matrix representation is as in lemma 3.16.

Let $\mathcal{V} = \mathbb{F}^4$ be a four dimensional vector space over a field $\mathbb{F}$ of characteristic unequal to two and $\tau : \mathcal{V}^* \to \mathcal{V}$ a linear map such that its matrix representation is as in lemma 3.16. According to lemma 3.17 the vector space $\mathcal{V}$ with skew-symmetric 3-linear map $[\cdot,\cdot,\cdot] = \tau \circ \phi\,(\cdot \wedge \cdot \wedge \cdot)$ is a 3-Lie algebra if and only if the matrix representation of τ satisfies

$$\begin{aligned}
(\tau_{34}-\tau_{43})\,\tau_{i2} + (\tau_{42}-\tau_{24})\,\tau_{i3} + (\tau_{23}-\tau_{32})\,\tau_{i4} &= 0\\
(\tau_{43}-\tau_{34})\,\tau_{i1} + (\tau_{14}-\tau_{41})\,\tau_{i3} + (\tau_{31}-\tau_{13})\,\tau_{i4} &= 0\\
(\tau_{24}-\tau_{42})\,\tau_{i1} + (\tau_{41}-\tau_{14})\,\tau_{i2} + (\tau_{12}-\tau_{21})\,\tau_{i4} &= 0\\
(\tau_{23}-\tau_{32})\,\tau_{i1} + (\tau_{31}-\tau_{13})\,\tau_{i2} + (\tau_{12}-\tau_{21})\,\tau_{i3} &= 0
\end{aligned}$$

for $i = 1,2,3,4$. Since the characteristic of $\mathbb{F}$ is unequal to two and the special shape of the matrix representation of τ, then the above equations are equivalent to

$$\begin{aligned}
\beta_2\tau_{i2} &= 0\\
\beta_2\tau_{i1} &= 0\\
\beta_1\tau_{i4} &= 0\\
\beta_1\tau_{i3} &= 0
\end{aligned}$$

for $i = 1,2,3,4$. After applying the constraints on the scalars $\alpha_i, \beta_i, \gamma_i \in \mathbb{F}$, the following equivalent equations are derived

$$\begin{aligned}
\alpha_3\beta_1 &= 0\\
\beta_2 &= 0\\
\gamma_i &= 0
\end{aligned}$$

for $i = 1,2,3,4$, and

$$\alpha_1 = 0 \quad\Rightarrow\quad \alpha_2 = 0 \quad\Rightarrow\quad \alpha_3 = 0 \quad\Rightarrow\quad \alpha_4 = 0.$$

Thus, the vector space $\mathcal{V}$ with a 3-linear map induced by a linear map $\tau : \mathcal{V}^* \to \mathcal{V}$ with a matrix representation as in lemma 3.16 is a 3-Lie algebra if and only if the matrix representation of τ satisfies

$$[\tau] = \begin{pmatrix} \alpha_1 & \beta & 0 & 0\\ -\beta & \alpha_2 & 0 & 0\\ 0 & 0 & \alpha_3 & 0\\ 0 & 0 & 0 & \alpha_4 \end{pmatrix}$$

with scalars $\alpha_i, \beta \in \mathbb{F}$ such that $\alpha_3\beta = 0$ and

$$\alpha_1 = 0 \quad\Rightarrow\quad \alpha_2 = 0 \quad\Rightarrow\quad \alpha_3 = 0 \quad\Rightarrow\quad \alpha_4 = 0.$$

The matrix representation of τ shows us that some 3-Lie algebras can not be isomorphic. At this point we can distinct three types of 3-Lie algebras:

- If $[\tau]$ is symmetric, then $\beta = 0$. The map $[\cdot,\cdot,\cdot]$ induced by τ is given by

$$\begin{aligned}
[e_1,e_2,e_3] &= {\alpha_4}'e_4\\
[e_1,e_2,e_4] &= {\alpha_3}'e_3\\
[e_1,e_3,e_4] &= {\alpha_2}'e_2\\
[e_2,e_3,e_4] &= {\alpha_1}'e_1
\end{aligned}$$

  with ${\alpha_i}' := (-1)^i\,\alpha_i$ for $i = 1,2,3,4$ such that

$${\alpha_1}' = 0 \quad\Rightarrow\quad {\alpha_2}' = 0 \quad\Rightarrow\quad {\alpha_3}' = 0 \quad\Rightarrow\quad {\alpha_4}' = 0.$$

- If $[\tau]$ is not symmetric and the rank of τ is two, then $\alpha_3 = \alpha_4 = 0$ and $\beta \neq 0$. The linear map $\mathcal{V} \to \mathcal{V}$ with matrix representation
$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \beta^{-1} & 0 & 0 \\ 0 & 0 & \beta & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$
is an isomorphism of 3-Lie algebras. This map transforms τ such that $\beta = 1$. The map $[\cdot,\cdot,\cdot]$ induced by τ is given by
$$\begin{aligned} [e_1, e_2, e_3] &= 0 \\ [e_1, e_2, e_4] &= 0 \\ [e_1, e_3, e_4] &= e_1 + \alpha_2 e_2 \\ [e_2, e_3, e_4] &= e_2 - \alpha_1 e_1 \end{aligned}$$
for scalars $\alpha_1, \alpha_2 \in \mathbb{F}$ such that $\alpha_1\alpha_2 + 1 \neq 0$ and
$$\alpha_1 = 0 \quad \Rightarrow \quad \alpha_2 = 0.$$
- If $[\tau]$ is not symmetric and the rank of τ is one, then $\alpha_3 = \alpha_4 = 0$, $\beta \neq 0$ and $\alpha_1\alpha_2 + \beta = 0$. The linear map $\mathcal{V} \to \mathcal{V}$ with matrix representation
$$\begin{pmatrix} \frac{1}{2\beta} & \frac{1}{2\alpha_2} & 0 & 0 \\ 0 & 0 & 2\beta & 0 \\ 0 & 0 & 0 & 1 \\ \frac{\beta}{2\alpha_1} & \frac{1}{2} & 0 & 0 \end{pmatrix}$$
is an isomorphism of 3-Lie algebras. The map $[\cdot,\cdot,\cdot]$ induced by τ is given by
$$\begin{aligned} [e_1, e_2, e_3] &= e_1 \\ [e_1, e_2, e_4] &= 0 \\ [e_1, e_3, e_4] &= 0 \\ [e_2, e_3, e_4] &= 0. \end{aligned}$$

Combining the successive parts completes the proof of theorem 3.13. Any four dimensional 3-Lie algebra over a field $\mathbb{F}$ is isomorphic to some 3-Lie algebra defined on the vector space $\mathbb{F}^4$. Every such a 3-Lie algebra is isomorphic to one of the three types of 3-Lie algebras given above. □

Corollary 3.18. *All four dimensional 3-Lie algebras over a field $\mathbb{F}$ of characteristic unequal to two which is algebraically closed are isomorphic to one of the following 3-Lie algebras. Furthermore no two of the 3-Lie algebras below are isomorphic to each other.*

- *The vector space $\mathbb{F}^4$ with basis $\{e_1, e_2, e_3, e_4\}$, and 3-linear map given by*
$$\begin{aligned} [e_1, e_2, e_3] &= \alpha_4 e_4 \\ [e_1, e_2, e_4] &= \alpha_3 e_3 \\ [e_1, e_3, e_4] &= \alpha_2 e_2 \\ [e_2, e_3, e_4] &= \alpha_1 e_1 \end{aligned}$$

with $\alpha_1, \alpha_2, \alpha_3, \alpha_4 \in \{0, 1\}$ *such that*

$$\alpha_1 = 0 \quad \Rightarrow \quad \alpha_2 = 0 \quad \Rightarrow \quad \alpha_3 = 0 \quad \Rightarrow \quad \alpha_4 = 0.$$

- *The vector space* $\mathbb{F}^4$ *with basis* $\{e_1, e_2, e_3, e_4\}$, *and 3-linear map given by*

$$\begin{aligned} [e_1, e_2, e_3] &= e_1 \\ [e_1, e_2, e_4] &= 0 \\ [e_1, e_3, e_4] &= 0 \\ [e_2, e_3, e_4] &= 0. \end{aligned}$$

- *The vector space* $\mathbb{F}^4$ *with basis* $\{e_1, e_2, e_3, e_4\}$, *and 3-linear map given by*

$$\begin{aligned} [e_1, e_2, e_3] &= 0 \\ [e_1, e_2, e_4] &= 0 \\ [e_1, e_3, e_4] &= e_1 \\ [e_2, e_3, e_4] &= e_2. \end{aligned}$$

- *The vector space* $\mathbb{F}^4$ *with basis* $\{e_1, e_2, e_3, e_4\}$, *and 3-linear map given by*

$$\begin{aligned} [e_1, e_2, e_3] &= 0 \\ [e_1, e_2, e_4] &= 0 \\ [e_1, e_3, e_4] &= e_1 + \alpha e_2 \\ [e_2, e_3, e_4] &= e_1 + e_2 \end{aligned}$$

with $\alpha \in \mathbb{F}$ *such that* $\alpha \neq 1$.

Proof. This proof extends the proof of theorem 3.13. Take the situation as at the end of that proof and let $\mathbb{F}$ be algebraically closed.

In lemma 3.15 we saw that two four dimensional 3-Lie algebras on the same vector space $\mathcal{V}$ are isomorphic if and only if there exists a bijective linear map $T : \mathcal{V} \to \mathcal{V}$ such that the relation

$$T \circ \tau_1 \circ T^* = \det(T)\, \tau_2$$

holds. Define $S := \lambda T$ for a nonzero scalar $\lambda \in \mathbb{F}$. Then the above relation is equivalent to

$$S \circ \tau_1 \circ S^* = \lambda^2 \det(T)\, \tau_2 = \frac{\det(S)}{\lambda^2} \tau_2.$$

The field $\mathbb{F}$ is algebraically closed and $\det(T) \neq 0$. So there exists a nonzero scalar λ such that $\lambda^2 \det(T) = 1$. Thus, two 3-Lie algebras on the same vector space $\mathcal{V}$ of dimension four are isomorphic if and only if there exists a bijective linear map $T : \mathcal{V} \to \mathcal{V}$ such that the relation

$$T \circ \tau_1 \circ T^* = \tau_2$$

holds. In matrix representation this relation is the same as the relation between congruent matrices.

Let $\tau : \mathcal{V}^* \to \mathcal{V}$ be the map corresponding to a 3-Lie algebra on the vector space $\mathcal{V} = \mathbb{F}^4$. Suppose the matrix representation of τ is given by

$$[\tau] = \begin{pmatrix} \alpha_1 & 0 & 0 & 0 \\ 0 & \alpha_2 & 0 & 0 \\ 0 & 0 & \alpha_3 & 0 \\ 0 & 0 & 0 & \alpha_4 \end{pmatrix}$$

and let $T : \mathcal{V} \to \mathcal{V}$ be a linear map with matrix representation

$$[T] = \begin{pmatrix} \lambda_1 & 0 & 0 & 0 \\ 0 & \lambda_2 & 0 & 0 \\ 0 & 0 & \lambda_3 & 0 \\ 0 & 0 & 0 & \lambda_4 \end{pmatrix}$$

for scalars λ_i such that if $\alpha_i \neq 0$, then $\lambda_i^2 \alpha_i = (-1)^i$, otherwise $\lambda_i = 1$. The linear map T is invertible. So the above 3-Lie algebra is isomorphic to

$$\begin{aligned} [e_1, e_2, e_3] &= {\alpha_4}' e_4 \\ [e_1, e_2, e_4] &= {\alpha_3}' e_3 \\ [e_1, e_3, e_4] &= {\alpha_2}' e_2 \\ [e_2, e_3, e_4] &= {\alpha_1}' e_1 \end{aligned}$$

with ${\alpha_1}', {\alpha_2}', {\alpha_3}', {\alpha_4}' \in \{0, 1\}$ such that

$${\alpha_1}' = 0 \quad \Rightarrow \quad {\alpha_2}' = 0 \quad \Rightarrow \quad {\alpha_3}' = 0 \quad \Rightarrow \quad {\alpha_4}' = 0.$$

Again let $\tau : \mathcal{V}^* \to \mathcal{V}$ be the map corresponding to a 3-Lie algebra on the vector space $\mathcal{V} = \mathbb{F}^4$. Suppose τ has a matrix representation

$$[\tau] = \begin{pmatrix} \alpha_1 & 1 & 0 & 0 \\ -1 & \alpha_2 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

for scalars $\alpha_1, \alpha_2 \in \mathbb{F}$ such that $\alpha_1 \alpha_2 + 1 \neq 0$ and

$$\alpha_1 = 0 \quad \Rightarrow \quad \alpha_2 = 0.$$

The third type of 3-Lie algebra in the corollary corresponds to the case $\alpha_1 = 0$. Assume $\alpha_1 \neq 0$. Let $T : \mathcal{V} \to \mathcal{V}$ be a linear map with matrix representation

$$[T] = \begin{pmatrix} \lambda_1 & 0 & 0 & 0 \\ 0 & \lambda_2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

for scalars λ_i such that $\alpha_1 {\lambda_1}^2 = -1$ and $\lambda_1 \lambda_2 = 1$. Then

$$[T \circ \tau \circ T^*] = \begin{pmatrix} -1 & 1 & 0 & 0 \\ -1 & -\alpha_1 \alpha_2 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

So the 3-Lie algebra corresponding to τ is isomorphic to

$$\begin{aligned}
[e_1, e_2, e_3] &= 0 \\
[e_1, e_2, e_4] &= 0 \\
[e_1, e_3, e_4] &= e_1 + \alpha e_2 \\
[e_2, e_3, e_4] &= e_1 + e_2
\end{aligned}$$

with $\alpha \in \mathbb{F}$ such that $\alpha \neq 1$.

To see that no two of the 3-Lie algebras above are isomorphic to each other, consider the rank of the matrix representation of $\tau : \mathcal{V} \to \mathcal{V}$ corresponding to each 3-Lie algebra and the rank of the symmetric and skew-symmetric parts thereof. This works for all types of 3-Lie algebras but the last. We prove by contradiction that also all 3-Lie algebra of the last type are distinct.

Assume that two 3-Lie algebras of the last type, with corresponding scalars α and β such that $\alpha \neq \beta$, are isomorphic. Then there exists an invertible 4×4 matrix Q such that

$$Q \begin{pmatrix} -1 & 1 & 0 & 0 \\ -1 & \alpha & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} Q^T = \begin{pmatrix} -1 & 1 & 0 & 0 \\ -1 & \beta & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

and more specifically, there exists a 2×2 matrix P such that

$$P \begin{pmatrix} -1 & 1 \\ -1 & \alpha \end{pmatrix} P^T = \begin{pmatrix} -1 & 1 \\ -1 & \beta \end{pmatrix}.$$

Then the symmetric and skew-symmetric parts of the above matrices satisfy

$$\begin{aligned}
P \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} P^T &= \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \\
P \begin{pmatrix} -1 & 0 \\ 0 & \alpha \end{pmatrix} P^T &= \begin{pmatrix} -1 & 0 \\ 0 & \beta \end{pmatrix}.
\end{aligned}$$

Taking the determinant of the above matrices gives equations $\det(P)^2 = 1$ and $\det(P)^2 \alpha = \beta$. These equations imply $\alpha = \beta$. This contradicts our assumption. Thus also the 3-Lie algebras of the last type are distinct. □

Remark 3.19. In the above corollary no two 3-Lie algebras are isomorphic to each other. Indeed there are almost as many 3-Lie algebras of the last type as there are elements in the field $\mathbb{F}$.

3.3 Quaternions

On a quaternion algebra the 3-linear map at the beginning of this chapter induces a four dimensional 3-Lie algebra. First we recall the definition of the quaternion algebra [17].

Definition 3.20. Let $\mathbb{F}$ be a field and let $\alpha, \beta \in \mathbb{F}$ be nonzero. The *quaternion algebra* is the set of elements $a+bi+cj+dk$ with $a, b, c, d \in \mathbb{F}$ called *quaternions*, component wise addition and multiplication determined by $ij = -ji = k$, $i^2 = \alpha$ and $j^2 = \beta$.

We define a skew-symmetric 3-linear map as

$$[x, y, z] := xyz + yzx + zxy - yxz - xzy - zyx$$

for quaternions x, y, z. Then

$$\begin{aligned}
[1, i, j] &= 2k \\
[1, i, k] &= 2\alpha j \\
[1, j, k] &= -2\beta i \\
[i, j, k] &= -6\alpha\beta.
\end{aligned}$$

If we view the quaternion algebra as a four dimensional vector space with basis $\{1, i, j, k\}$, then the above 3-linear map on this vector space clearly is one of the types in theorem 3.13. Thus a quaternion algebra induces a four dimensional 3-Lie algebra.

Over the real numbers $\mathbb{R}$ there are two distinct quaternion algebras up to an isomorphism. For $\alpha = \beta = -1$ we have Hamilton's quaternions, and for $\alpha = \beta = 1$ the quaternion algebra can be identified with the 2×2 matrices. This explains why the Jacobi identity is satisfied for real 2×2 matrices.

Theorem 3.21. *If two quaternion algebras are isomorphic, then also their corresponding 3-Lie algebras are isomorphic.*

Proof. This follows from the fact that isomorphisms respect addition and multiplication. □

It is not known whether the reverse statement is also true.

3.4 Octonions

The quaternions are a generalization of the complex numbers. In the same way octonions generalize quaternions. The octonions are no longer associative, but it is an alternative algebra [15], that is, it satisfies

$$\begin{aligned}
(xy)\,x &= x\,(yx) \\
x\,(xy) &= x^2 y \\
(xy)\,y &= xy^2
\end{aligned}$$

for all elements x, y.

A natural way to define a 3-linear map is to use the associator

$$[x, y, z] = (xy)\,z - x\,(yz)$$

for all elements x, y, z. We can prove this is a skew-symmetric map using the above relations. For example

$$[x, x, y] = x^2 y - x\,(xy) = 0$$

so we have

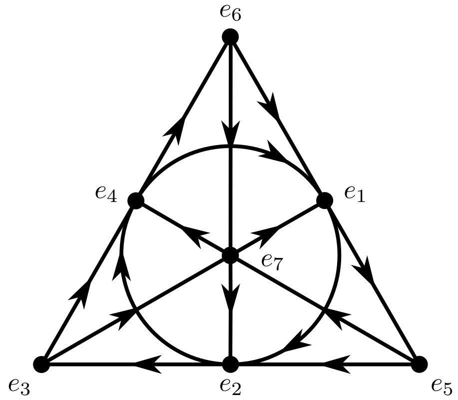
$$0 = [x + y, x + y, z] = [x, y, z] + [y, x, z]$$

for all elements x, y, z. The other cases are proven in the same way. Thus the associator is skew-symmetric. However this 3-linear map does not satisfy the Jacobi identity. Before we can show this, we need to describe how to multiply octonions.

We write an octonion x as

$$x = a_0 + a_1e_1 + a_2e_2 + a_3e_3 + a_4e_4 + a_5e_5 + a_6e_6 + a_7e_7$$

for $a_0, \ldots, a_7 \in \mathbb{R}$, where $\{1, e_1, e_2, e_3, e_4, e_5, e_6, e_7\}$ is a basis of the octonion algebra as in [1]. To describe the multiplication of octonions, we note that 1 is the identity and ${e_i}^2 = -1$ for all $i = 1, \ldots, 7$. All other products can be resolved from the *Fano plane*, which is given in the figure below.



Each line passes through three points. If e_i, e_j, e_k are three such points ordered according to the arrows on the line, then $e_ie_j = e_k$ and $e_je_i = -e_k$. For example $e_5e_2 = e_3$ and $e_3e_2 = -e_5$.

Now we show that the Jacobi identity is not satisfied. Lets check it for the vectors e_1, e_2, e_4, e_1, e_7. Then

$$\begin{aligned}
[e_1, e_2, e_4] &= 0 \\
[e_1, e_1, e_7] &= 0 \\
[e_2, e_1, e_7] &= 2e_5 \\
[e_4, e_1, e_7] &= -2e_6
\end{aligned}$$

and

$$\begin{aligned}
[e_1, [e_2, e_1, e_7], e_4] &= -4e_3 \\
[e_1, e_2, [e_4, e_1, e_7]] &= -4e_3.
\end{aligned}$$

So we have

$$\begin{aligned}
&[[e_1, e_2, e_4], e_1, e_7] = \\
&\qquad [[e_1, e_1, e_7], e_2, e_4] + [e_1, [e_2, e_1, e_7], e_4] + [e_1, e_2, [e_4, e_1, e_7]] + 8e_3
\end{aligned}$$

that is, the Jacobi identity is not satisfied.

Using the associator as a skew-symmetric 3-linear map on the octonions does not give a 3-Lie algebra. However it is possible to construct a generalized 3-Lie algebra using the octonions as is shown in [18].

Chapter 4

Conclusions

In chapter 2 the definition of a n-Lie algebra was given with corresponding definitions such as subalgebras, ideals and isomorphisms. We saw an infinite dimensional n-Lie algebra in example 2.12. A new m-Lie algebra can be constructed from a n-Lie algebra for $m < n$ as we saw in theorems 2.14 and 2.15.

We classified the 3-Lie algebras of low dimensions in chapter 3. There were only trivial 3-Lie algebras of dimension up to three, one nontrivial 3-Lie algebra of dimension three, and many nontrivial four dimensional 3-Lie algebras as we saw in theorems 3.11, 3.12 and 3.13 respectively. In general the 3-linear map

$$[A, B, C] := ABC + BCA + CAB - BAC - ACB - CBA$$

for matrices A, B, C does not satisfy the Jacobi identity as we saw in example 3.1, but in section 3.3 we saw that on the quaternion algebra it does define a 3-Lie algebra. Furthermore the octonions with the associator as the 3-linear map is not a 3-Lie algebra.

Appendix A

Congruent matrices

From linear algebra we know that every symmetric matrix is congruent to a diagonal matrix [8]. In this appendix we prove a similar result for any square matrix over a field of characteristic unequal to two. Let us first recall the definition of congruent matrices [8].

Definition A.1. Let A and B be $n \times n$ matrices. Then B is said to be *congruent* to A if there exists an invertible $n \times n$ matrix Q such that $B = Q^T AQ$.

The first theorem we prove is a statement about the vector space on which a skew-symmetric bilinear form is defined. The vector space can be split into a number of two dimensional subspaces such that the bilinear form is nonzero, and one remaining subspace on which the bilinear form is zero. The proof of this theorem is inspired by the proof of theorem 6.35 in [8] about diagonalizable symmetric bilinear forms.

Theorem A.2. *Let $B : \mathcal{V} \times \mathcal{V} \to \mathbb{F}$ be a skew-symmetric bilinear form on a vector space $\mathcal{V}$ of finite dimension n over a field $\mathbb{F}$ of characteristic unequal to two. Then there exists a nonnegative integer $k \leq \frac{n}{2}$ and two dimensional subspaces $\mathcal{V}_1, \ldots, \mathcal{V}_k$ of $\mathcal{V}$ and a $n - 2k$ dimensional subspace $\mathcal{V}_0$ of $\mathcal{V}$ such that*

$$\mathcal{V} = \mathcal{V}_0 \oplus \cdots \oplus \mathcal{V}_k$$

with B restricted to $\mathcal{V}_0$ is zero, B restricted to $\mathcal{V}_i$ is nonzero for $i = 1, \ldots, k$ and $B(x, y) = 0$ for all $x \in \mathcal{V}_i$, $y \in \mathcal{V}_j$ with $i \neq j$.

Proof. If $B(x, y) = 0$ for all $x, y \in \mathcal{V}$, then choose $k = 0$ and define $\mathcal{V}_0 := \mathcal{V}$. This proves the theorem in the case B is a zero bilinear form.

If $n = 0$, then $B(x, y) = 0$ by bilinearity. Suppose $n = 1$. Let $x, y \in \mathcal{V}$ be any pair of vectors. If x or y or both are zero vectors, then $B(x, y) = 0$ by bilinearity. If x and y are both nonzero vectors, then there exists a nonzero scalar $\lambda \in \mathbb{F}$ such that $x = \lambda y$. Again $B(x, y) = 0$ by skew-symmetry as follows from

$$B(x, y) = \lambda B(x, x) = -\lambda B(x, x) = -B(x, y).$$

Thus if $n = 0$ or $n = 1$, then $B(x, y) = 0$ for all $x, y \in \mathcal{V}$. Combining this result with the above statement completes the proof for $n = 0$ and $n = 1$.

Assume B is nonzero and the theorem holds for $n-2$. There exists a pair of nonzero vectors $x, y \in \mathcal{V}$ and a nonzero scalar $\lambda \in \mathbb{F}$ such that $B(x,y) = \lambda$. Define a linear map $L : \mathcal{V} \to \mathbb{F}^2$ as

$$L_{x,y}(z) := \begin{pmatrix} B(z,y) \\ B(x,z) \end{pmatrix}$$

for any $z \in \mathcal{V}$. The map L is linear, because B is bilinear. The rank of the map L is two, because the rank can at most be two and

$$L_{x,y}(x) = \begin{pmatrix} \lambda \\ 0 \end{pmatrix} \quad \text{and} \quad L_{x,y}(y) = \begin{pmatrix} 0 \\ \lambda \end{pmatrix}$$

requires the rank to be at least two.

Define $\mathcal{V}' := \ker L_{x,y}$. The dimension theorem implies that $\dim \mathcal{V}' = n-2$. By assumption the theorem can be applied to the restriction of B to $\mathcal{V}'$. There exists a nonnegative integer $k' \leq \frac{n-2}{2}$ and two dimensional subspaces $\mathcal{V}_1, \ldots, \mathcal{V}_{k'}$ of $\mathcal{V}'$ and a $n-2(k'+1)$ dimensional subspace $\mathcal{V}_0$ of $\mathcal{V}'$. Define $k := k'+1$ and $V_k := \operatorname{span}\{x,y\}$. Then

$$\mathcal{V} = \mathcal{V}' \oplus \mathcal{V}_k = \mathcal{V}_0 \oplus \cdots \oplus \mathcal{V}_{k'} \oplus \mathcal{V}_k.$$

The construction of the linear map L and subspace $\mathcal{V}'$ implies that B restricted to $\mathcal{V}_k$ is nonzero and $B(x,y) = 0$ for all $x \in \mathcal{V}'$ and $y \in \mathcal{V}_k$. Thus also that $B(x,y) = 0$ for all $x \in \mathcal{V}_i$ and $y \in \mathcal{V}_k$ with $i = 1, \ldots, k-1$. If this is combined with the results for $\mathcal{V}'$, then the theorem is also proved for n.

By induction the theorem holds for any nonnegative integer n. $\square$

The space of bilinear forms on a vector space of dimension n is isomorphic to the vector space of $n \times n$ matrices. Combining this statement with the above theorem gives us a simple statement about congruent skew-symmetric matrices.

Corollary A.3. *Let A be a skew-symmetric $n \times n$ matrix over a field $\mathbb{F}$ of characteristic unequal to two. Then A has even rank and is congruent to*

$$\begin{pmatrix} D_1 & O & \cdots & O \\ O & D_2 & \cdots & O \\ \vdots & \vdots & \ddots & \vdots \\ O & O & \cdots & D_{k+1} \end{pmatrix} \tag{A.1}$$

with $2k = \operatorname{rank} A$, D_{k+1} is the $n-2k \times n-2k$ zero matrix and for $i = 1, \ldots, k$

$$D_i = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

Proof. Let $\mathcal{V}$ be any n dimensional vector space over the field $\mathbb{F}$ with an ordered basis β. There exists a bilinear form $B : \mathcal{V} \times \mathcal{V} \to \mathbb{F}$ such that A is the matrix representation of B with respect to β. The matrix A is skew-symmetric, thus the bilinear form B is also skew-symmetric. Apply theorem A.2 to B and let k, $\mathcal{V}_i$ for $i = 0, \ldots, k$ be as in the theorem.

Suppose $1 \leq i \leq k$. There exists nonzero vectors $x, y \in \mathcal{V}_i$ and a nonzero scalar $\lambda \in \mathbb{F}$ such that $B(x,y) = \lambda$, because B restricted to $\mathcal{V}_i$ is nonzero.

Define $u_i := x$ and $v_i := \lambda^{-1} y$. The skew-symmetry of B implies the linear independence of the vectors x and y, and hence also of u_i and v_i. The dimension of $\mathcal{V}_i$ is two. Therefore $\{u_i, v_i\}$ is an ordered basis of $\mathcal{V}_i$. Furthermore the matrix representation of B restricted to $\mathcal{V}_i$ is D_i.

Theorem A.2 states that B restricted to $\mathcal{V}_0$ is zero and $\dim \mathcal{V}_0 = n - 2k$. Let $\{w_1, \ldots, w_{n-2k}\}$ be any ordered basis of $\mathcal{V}_0$. The matrix representation of B restricted to $\mathcal{V}_0$ is the $n - 2k \times n - 2k$ zero matrix, that is D_{k+1}.

Define the ordered set $\gamma := \{u_1, v_1, \ldots, u_k, v_k, w_1, \ldots, w_{n-2k}\}$. The set γ consists of n linear independent vectors. That is, γ is an ordered basis of $\mathcal{V}$. The matrix representation of B with respect to γ is as in equation (A.1), because $B(x, y) = 0$ for all $x \in \mathcal{V}_i$, $y \in \mathcal{V}_j$ with $i \neq j$. This proves A is congruent to the matrix in (A.1).

The rank of A is a direct consequence of the shape of the matrix in (A.1). $\square$

In theorem A.2 the vector space is split up into a number of subspaces. The proof of the above corollary illustrates there is some freedom left in choosing a basis for the subspaces. This is exploited in the proof of the following theorem. This theorem is a statement about any square matrix.

Theorem A.4. *Let A be a $n \times n$ matrix over a field $\mathbb{F}$ of characteristic unequal to two. Then the matrix is congruent to*

$$\begin{pmatrix} D_1 & C_{12} & \cdots & C_{1k'} \\ {C_{12}}^T & D_2 & \cdots & C_{2k'} \\ \vdots & \vdots & \ddots & \vdots \\ {C_{1k'}}^T & {C_{2k'}}^T & \cdots & D_{k'} \end{pmatrix} \tag{A.2}$$

with $k' = k+1$, the rank of the skew-symmetric part of A is $2k$, $D_{k'}$ is a diagonal matrix and for $i = 1, \ldots, k$

$$D_i = \begin{pmatrix} \alpha_i & 1 \\ -1 & \beta_i \end{pmatrix}$$

for some $\alpha_i, \beta_i \in \mathbb{F}$.

Proof. Let $\mathcal{V}$ be a n dimensional vector space over the field $\mathbb{F}$ with an ordered basis γ. There exists a bilinear form $B : \mathcal{V} \times \mathcal{V} \to \mathbb{F}$ such that A is the matrix representation of B with respect to γ. Define new bilinear forms $B_S : \mathcal{V} \times \mathcal{V} \to \mathbb{F}$ and $B_A : \mathcal{V} \times \mathcal{V} \to \mathbb{F}$ as

$$\begin{aligned} B_S(x, y) &:= \frac{B(x, y) + B(y, x)}{2} \\ B_A(x, y) &:= \frac{B(x, y) - B(y, x)}{2} \end{aligned}$$

for all $x, y \in \mathcal{V}$. Then B_S is a symmetric bilinear form, B_A is a skew-symmetric bilinear form and $B = B_S + B_A$.

Apply theorem A.2 to B_A. Let k and $\mathcal{V}_i$ for $i = 0, \ldots, k$ be as in that theorem. The restriction of B_S to $\mathcal{V}_i$ is again a symmetric bilinear form. Thus there exists an ordered basis δ_i of $\mathcal{V}_i$ such that the restriction of B_S is diagonal.

Suppose $1 \leq i \leq k$. The subspace $\mathcal{V}_i$ has dimension two. The basis δ_i contains two vectors, that is $\delta_i = \{x_i, y_i\}$. The restriction of B_A to $\mathcal{V}_i$ is nonzero. There exists a pair of vectors $u, v \in \mathcal{V}_i$ and a nonzero scalar $\lambda \in \mathbb{F}$ such that $B_A(u,v) = \lambda$. Expressing the vectors u, v in terms of the ordered basis gives $u = u_1 x_i + u_2 y_i$ and $v = v_1 x_i + v_2 y_i$. Define $a_i := \lambda^{-1} x_i$ and $b_i := (u_1 v_2 - u_2 v_1)\, y_i$. Then $\{a_i, b_i\}$ is again an ordered basis of $\mathcal{V}_i$ and the matrix representation of B with respect to this ordered basis is D_i for some scalars α_i and β_i.

The restriction of B_A to $\mathcal{V}_0$ is zero. The dimension of $\mathcal{V}_0$ is equal to $n - 2k$, that is, the ordered basis $\delta_0 = \{c_1, \ldots, c_{n-2k}\}$. The matrix representation of B with respect to δ_0 is a diagonal matrix. Denote this matrix by $D_{k'}$.

Define the ordered set $\gamma' = \{a_1, b_1, \ldots, a_k, b_k, c_1, \ldots, c_{n-2k}\}$. This set contains n linear independent vectors, that is, the set γ' is an ordered basis of $\mathcal{V}$. For all $x \in \mathcal{V}_i$, $y \in \mathcal{V}_j$ with $i \neq j$ the bilinear form $B_A(x,y) = 0$, that is, $B(x,y) = B_S(x,y)$. Combining these results shows that the matrix representation of B with respect to γ' is given by the matrix in A.2. This completes the proof. $\square$

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